More about Lagrange multiplier

<u>Question</u> : Finding $\min f(x,y)$ or $\max f(x,y)$

given $g(x,y) = C$

Define $F(\lambda, c, x, y) = f(x,y) - \lambda[g(x,y) - c]$

<u>Method</u> : ① find critical point of $F(\lambda, x, y)$

$$\begin{cases} \dfrac{\partial F}{\partial x} = f_x - \lambda g_x = 0 \\ \dfrac{\partial F}{\partial y} = f_y - \lambda g_y = 0 \\ \dfrac{\partial F}{\partial \lambda} = g(x,y) - c = 0 \end{cases} \Rightarrow \text{solve } (\lambda, x, y)$$

② What is λ ? $\dfrac{\partial F}{\partial c} = \lambda$

<u>Economics</u> max utility function $u(x,y) = x^\alpha y^{1-\alpha}$, x = labor, y = capital, $0 < \alpha < 1$

Consumer budget (wealth) = W

price P_x (for x) P_y (for y)

$$P_x x + P_y y = W$$

$$\begin{cases} \max x^\alpha y^{1-\alpha} \\ \text{given } P_x x + P_y y = W \end{cases}$$

For example

$$\begin{cases} \max x^{\frac{1}{4}} y^{\frac{2}{4}} \\ \text{given } 2x + 3y = 500 \end{cases}$$

$$F(\lambda, x, y) = x^\alpha y^{1-\alpha} - \lambda [P_x x + P_y y - W]$$

$$\frac{\partial F}{\partial \lambda} = P_x x + P_y y - W = 0$$

$$\frac{\partial F}{\partial x} = \alpha x^{\alpha-1} y^{1-\alpha} - \lambda P_x = 0 \overset{*x}{\Rightarrow} \alpha x^\alpha y^{1-\alpha} = \lambda P_x x$$

$$\frac{\partial F}{\partial y} = (1-\alpha) x^\alpha y^{-\alpha} - \lambda P_y = 0 \overset{*y}{\Rightarrow} (1-\alpha) x^\alpha y^{1-\alpha} = \lambda P_y y$$

$$+$$

$$\Rightarrow \quad x^\alpha y^{1-\alpha} = \lambda (P_x x + P_y y) = \lambda W$$

So when (λ^*, x^*, y^*) is the optimal value

$$\cancel{(x^*)^\alpha y^{1-\alpha}} \quad (x^*)^\alpha (y^*)^{1-\alpha} = \lambda^* W$$

λ^* = marginal utility of ~~wealth~~ budget $\qquad \lambda^* = \dfrac{(x^*)^\alpha (y^*)^{1-\alpha}}{W}$

(shadow price)

Solve the problem $\qquad x^\alpha y^{1-\alpha} = \lambda W$

$$\Rightarrow \alpha \lambda W = \lambda P_x x \Rightarrow P_x x = \alpha W \Rightarrow x = \frac{\alpha W}{P_x}$$

$$(1-\alpha)\lambda W = \lambda P_y y \Rightarrow P_y y = (1-\alpha) W \Rightarrow y = \frac{(1-\alpha) W}{P_y}$$

$$\lambda = \frac{\left(\frac{\alpha W}{P_x}\right)^\alpha \left(\frac{(1-\alpha) W}{P_y}\right)^{1-\alpha}}{W}$$

$$\lambda = \frac{\alpha^\alpha}{P_x^{\alpha}} \cdot \frac{(1-\alpha)^{1-\alpha}}{P_y^{1-\alpha}}$$