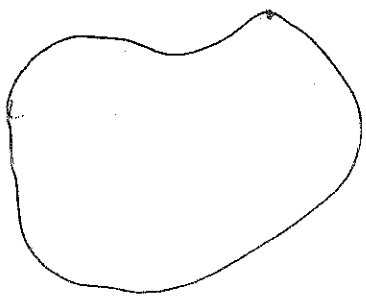
Math 212 Lecture 12

min/max $f(x,y)$ given $g(x,y)=K$.



$g(x(t),y(t)) = K$

$\Rightarrow g_x x' + g_y y' = 0$

$f(x(t),y(t)) = F(t)$

$F'(t) = f_x x' + f_y y' = 0$

$\Rightarrow$ critical pt.

So $\langle g_x, g_y\rangle \langle x', y'\rangle = 0$ always

$\langle f_x, f_y\rangle \langle x', y'\rangle = 0$ at critical pt of $F(t)$

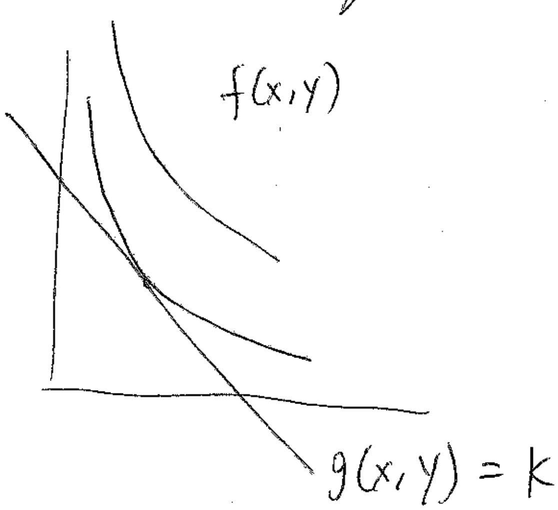
$\Rightarrow \langle g_x, g_y\rangle \parallel \langle f_x, f_y\rangle$ or $\nabla f = \lambda \nabla g$

Find extreme values of $f(x,y)$ given $g(x,y)=k$.

① Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = k \end{cases}$ $\quad \lambda$ = Lagrange multiplier

② Compare all solutions $\Rightarrow$ min, max,

geometric meaning



extreme values are achieved when the level curve of $f(x,y)$ is tangent to $g(x,y)=k$.

(1) $f(x,y) = 3x+y$ given $x^2+y^2=10$

$$g(x,y) = x^2+y^2-10$$

$$\nabla f = \langle 3, 1\rangle \qquad \nabla g = \langle 2x, 2y\rangle$$

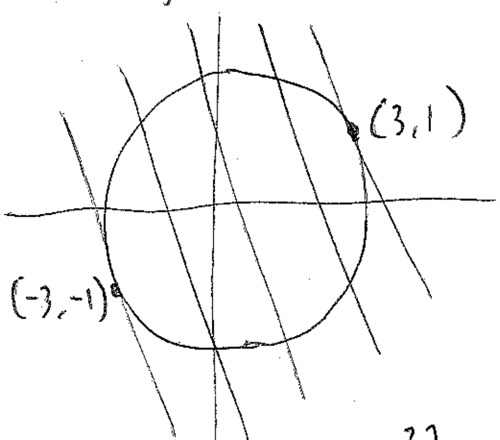
$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 3 = \lambda \cdot 2x \\ 1 = \lambda \cdot 2y \\ x^2+y^2=10 \end{cases} \Rightarrow \text{solve } (\lambda, x, y)$$

$$x = \frac{3}{2\lambda},\ y = \frac{1}{2\lambda} \Rightarrow \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 10 \Rightarrow \frac{10}{4\lambda^2} = 10$$

$$\Rightarrow 4\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{1}{2}$$

$\lambda = \frac{1}{2}$, $x=3$, $y=1$, $\quad \lambda = -\frac{1}{2}$, $x=-3$, $y=-1$

$f(3,1) = 10$, $\quad f(-3,-1) = -10$ $\quad$ So maximum $=10$ at $(x,y)=(3,1)$
minimum $=-10$ at $(x,y)=(-3,-1)$



(2) $f(x,y) = x^2+y^2$ given that $3x+y=10$

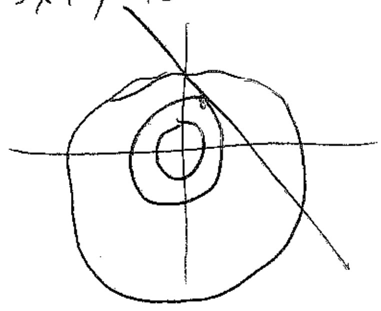
$$g(x,y) = 3x+y-10$$

$$\nabla f = \langle 2x, 2y\rangle \quad \nabla g = \langle 3, 1\rangle$$

$$\begin{cases} 2x = 3\lambda & \Rightarrow x = \frac{3\lambda}{2} \\ 2y = \lambda & \Rightarrow y = \frac{\lambda}{2} \Rightarrow 3\cdot\frac{3\lambda}{2} + \frac{\lambda}{2} = 10 \Rightarrow \frac{10\lambda}{2} = 10 \Rightarrow \lambda = 2 \Rightarrow \begin{matrix} x=3 \\ y=1 \end{matrix} \\ 3x+y=10 \end{cases}$$

$(x,y) = (3,1) \Rightarrow$ min $\quad$ no maximum since on $3x+y=10$

$$x^2+y^2 \longrightarrow \infty$$



① and ② are dual problems

③ $f(x,y) = 2x^2 + 3y^2 - 4x + 6y - 5$ on $D = \{(x,y) : x^2+y^2 \le 16\}$

Step 1 $f_x = 4x - 4 = 0$, $f_y = 6y + 6 = 0$ $\Rightarrow$ $x = 1$, $y = -1$ $\quad (1,-1) \in D$ since $1^2 + (-1)^2 = 2 < 16$

Step 2 Boundary case $f(x,y)$ with $x^2+y^2 = 16$ $\quad g(x,y) = x^2+y^2-16$

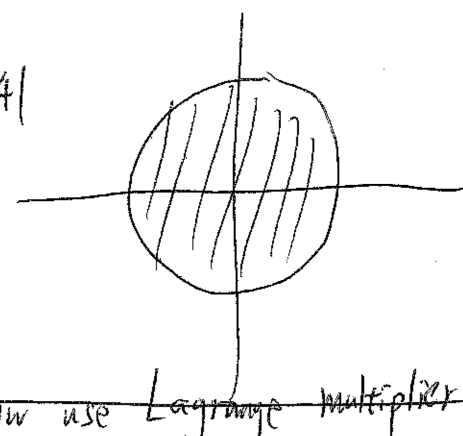
$$\nabla f = \langle 4x-4, 6y+6 \rangle = \lambda \nabla g = \lambda \langle 2x, 2y \rangle$$

$$\begin{cases} 4x-4 = 2\lambda x & \Rightarrow 2x - \lambda x = 2 \Rightarrow x = \frac{2}{2-\lambda} \\ 6y+6 = 2\lambda y & \Rightarrow 3y - \lambda y = -3 \Rightarrow y = \frac{-3}{3-\lambda} \\ x^2+y^2-16 = 0 \end{cases}$$

$$\left(\frac{2}{2-\lambda}\right)^2 + \left(\frac{-3}{3-\lambda}\right)^2 = 16 \Rightarrow 4(3-\lambda)^2 + 9(2-\lambda)^2 = 16(2-\lambda)^2(3-\lambda)^2$$

W—Alpha $\Rightarrow \lambda \approx 1.43078 \quad \lambda \approx 3.78141$

$\Rightarrow$ numerical solution



④ ~~We solved this one earlier, but now use Lagrange multiplier.~~

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} \quad \text{given } x - y + z = 4$$

$$f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2 \quad \text{given } x - y + z = 4$$

$f_x = 2(x-1)$ | $g_x = 1$
$f_y = 2(y-2)$ | $g_y = -1$ $\Rightarrow$
$f_z = 2(z-3)$ | $g_z = 1$

$$\begin{cases} 2(x-1) = \lambda \\ 2(y-2) = -\lambda \\ 2(z-3) = \lambda \\ x - y + z = 4 \end{cases}$$

$$\begin{cases} x = \frac{\lambda}{2} + 1 \\ y = \frac{-\lambda}{2} + 2 \\ z = \frac{\lambda}{2} + 3 \end{cases} \Rightarrow \left(\frac{\lambda}{2}+1\right) - \left(\frac{-\lambda}{2}+2\right) + \left(\frac{\lambda}{2}+3\right) = 4$$

$$\frac{3\lambda}{2} + 2 = 4 \Rightarrow \frac{3\lambda}{2} = 2 \Rightarrow \lambda = \frac{4}{3}.$$

$$\Rightarrow \begin{cases} x = \frac{5}{3} \\ y = \frac{4}{3} \\ z = \frac{11}{3} \end{cases} \qquad d = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2} = \frac{2}{3} \cdot \sqrt{3}$$

(5) $x^2+y^2+z^2=1$ to $(1,2,3)$

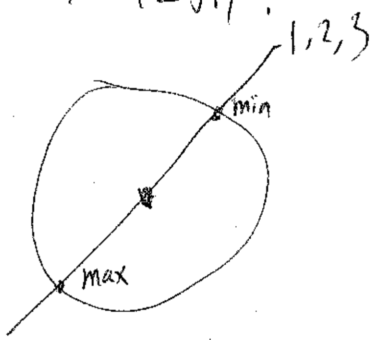
$d = \sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$ given $x^2+y^2+z^2=1$

$f(x,y,z) = (x-1)^2+(y-2)^2+(z-3)^2 \qquad g(x,y,z) = x^2+y^2+z^2-1$

$f_x = 2(x-1)$	$g_x = 2x$	$2(x-1) = 2\lambda x \Rightarrow x-1 = \lambda x \Rightarrow (1-\lambda)x = 1$, $x = \frac{1}{1-\lambda}$
$f_y = 2(y-2)$	$g_y = 2y$	$\Rightarrow 2(y-2) = 2\lambda y \Rightarrow y-2 = \lambda y \Rightarrow y = \frac{2}{1-\lambda}$
$f_z = 2(z-3)$	$g_z = 2z$	$2(z-3) = 2\lambda z$, $z = \frac{3}{1-\lambda}$

$$\left(\frac{1}{(1-\lambda)}\right)^2 + \left(\frac{2}{1-\lambda}\right)^2 + \left(\frac{3}{(1-\lambda)}\right)^2 = 1 \Rightarrow \frac{14}{(1-\lambda)^2} = 1 \Rightarrow (1-\lambda)^2 = 14$$

$\lambda = 1 \pm \sqrt{14}$. $\qquad (x,y,z) = \frac{1}{\sqrt{14}}(1,2,3)$ or $\frac{-1}{\sqrt{14}}(1,2,3)$



$$d_{min} = \sqrt{14\left(\frac{1}{\sqrt{14}}-1\right)^2} = \sqrt{14} \cdot \left(\frac{-1}{\sqrt{14}}+1\right) = \sqrt{14} - 1$$

$$d_{max} = \sqrt{14\left(-\frac{1}{\sqrt{14}}-1\right)^2} = \sqrt{14}\left(\frac{1}{\sqrt{14}}+1\right) = \sqrt{14} + 1$$