Math 212 Lecture 11

$z = f(x, y)$ Assume f, (f_x, f_y), $(f_{xx}, f_{xy}, f_{yx}, f_{yy})$ exist

Thm 1 If (x_0, y_0) is a local min or local max for $z = f(x,y)$, then

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

proof. (x_0, y_0) is a local min $\Rightarrow$ x_0 is a local min of $f(x, y_0)$

$\Rightarrow f_x(x_0, y_0) = 0$ and y_0 is a local min of $f(x_0, y)$

$\Rightarrow f_y(x_0, y_0) = 0$

Thm 2 The 2nd order directional derivative of $f(x,y)$ in direction $\langle a,b\rangle$ at (x_0, y_0) is

$$D_{uu} f = f_{xx} a^2 + 2 f_{xy} ab + f_{yy} b^2$$

If $f_{xx}(x_0, y_0) > 0$ and $D = f_{xx} f_{yy} - (f_{xy})^2 > 0$

then $f_{xx} a^2 + 2 f_{xy} ab + f_{yy} b^2 > 0$ for any (a,b) s.t. $a^2 + b^2 = 1$

So $D_{uu} f(x_0, y_0) \langle a,b\rangle > 0$, $D_u f(x_0, y_0)\langle a,b\rangle = 0$

$\Rightarrow$ f achieved local min in (a,b) direction for any direction (a,b)

$\Rightarrow$ f achieves a local min at (x_0, y_0)

local max $\Rightarrow$ Similar

Saddle pt? not proved in book, one can use Taylor expansion to prove.

Example 1 (1) $f(x,y) = x^3y + 12x^2 - 8y$

$f_x = 3x^2y + 24x = 0 \quad f_y = x^3 - 8 = 0$

$\Downarrow$

$3 \cdot 2^2 y + 24 \cdot 2 = 0 \qquad x = 2$

$y = -\frac{48}{12} = -4 \qquad \Rightarrow$ critical pt $(x,y) = (2,-4)$

$f_{xx} = 6xy + 24$, $f_{xy} = 3x^2 = f_{yx}$, $f_{yy} = 0 \quad H = \begin{pmatrix} -24 & 12 \\ 12 & 0 \end{pmatrix}$

$(2,-4) \Rightarrow f_{xx}(2,-4) = -24$, $f_{xy} = 12$, $f_{yy} = 0$

$D = f_{xx}f_{yy} - f_{xy}^2 = -12^2 = -144 < 0 \Rightarrow$ saddle pt. $(2,-4)$

(2) $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$

$f_x = 6x^2 + y^2 + 10x = 0 \quad f_y = 2xy + 2y = 0$

$$\begin{cases} 6x^2 + y^2 + 10x = 0 \\ 2xy + 2y = 0 \Rightarrow 2y(x+1) \Rightarrow y = 0 \text{ or } x = -1 \end{cases}$$

If $y = 0$, $6x^2 + 10x = 0 \Rightarrow x(6x+10) = 0 \Rightarrow x = 0$ or $x = -\frac{10}{6} = -\frac{5}{3}$,

If $x = -1$, $6 + y^2 + (-10) = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$

<u>Critical pts</u>: $(0,0)$, $(-\frac{5}{3}, 0)$, $(-1, 2)$, $(-1,-2)$

$f_{xx} = 12x + 10$, $f_{xy} = 2y = f_{yx}$, $f_{yy} = 2x + 2$

$H(0,0) = \begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{matrix} D = 20 > 0 \\ f_{xx} > 0, f_{yy} > 0 \end{matrix} \Rightarrow$ local min.

$H(-\frac{5}{3}, 0) = \begin{pmatrix} -10 & 0 \\ 0 & -\frac{4}{3} \end{pmatrix} \quad D = \frac{40}{3} > 0 \quad f_{xx} < 0, f_{yy} < 0 \Rightarrow$ local max

$$H(-1,2) = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix} \quad D = -16 < 0 \Rightarrow \text{saddle}.$$

$$H(-1,-2) = \begin{pmatrix} -2 & -4 \\ -4 & 0 \end{pmatrix} \quad D = -16 < 0 \Rightarrow \text{saddle}.$$

(3) Find the shortest distance from $(0,1,1)$ to $x-2y+3z=6$.

Take a pt (x,y,z) on $x-2y+3z=6$

$$\text{Distance} = \sqrt{(x-0)^2+(y-1)^2+(z-1)^2} = \sqrt{x^2+(y-1)^2+(z-1)^2}$$

Since (x,y,z) is on $x-2y+3z=6$, then $x = 6+2y-3z$

$$\Rightarrow \sqrt{(6+2y-3z)^2+(y-1)^2+(z-1)^2}$$

Distance = shortest $\Leftrightarrow$ Distance^2 = smallest

Let
$$f(y,z) = (6+2y-3z)^2+(y-1)^2+(z-1)^2$$
$$= 36 + 4y^2 + \boxed{9z^2} + 24y - 36z - 12yz + y^2 - 2y + 1 + \boxed{z^2} - 2z + 1$$
$$= 5y^2 + 10z^2 - 12yz + 22y - 38z + 38$$

$f_y = 10y - 12z + 22 = 0 \qquad 5y - 6z = -11 \quad \times 6 \Rightarrow 30y - 36z = -66$

$f_z = 20z - 12y - 38 = 0 \qquad -6y + 10z = 19 \quad \times 5 \Rightarrow -30y + 50z = 95$

$$\Rightarrow 14z = 29 \Rightarrow z = \frac{29}{14}, \quad y = \frac{6z-11}{5} = \frac{6\cdot\frac{29}{14} - \frac{154}{14}}{5} = \frac{20}{70} = \frac{4}{14} = \frac{2}{7}$$

$$(x,y,z) = \left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right) = \left(\frac{5}{14}, \frac{4}{14}, \frac{29}{14}\right)$$

$$x = 6+2y-3z = 6 + 2\cdot\frac{2}{7} - 3\cdot\frac{29}{14}$$
$$= 6 + \frac{4}{7} - \frac{87}{14} = \frac{5}{14}$$

$$\text{Distance} = \sqrt{\left(\frac{5}{14}\right)^2 + \left(\frac{10}{14}\right)^2 + \left(\frac{15}{14}\right)^2} = \frac{5\sqrt{1^2+2^2+3^2}}{14} = \frac{5}{\sqrt{14}}$$

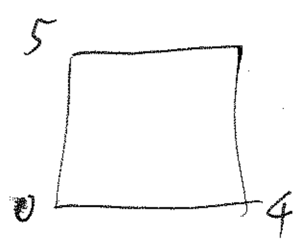
In Section 12.5

$$D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|1\cdot 0 - 2\cdot 1 + 3\cdot 1 - 6|}{\sqrt{1^2+2^2+3^2}} = \frac{5}{\sqrt{14}}$$

Example 2 (7) $f(x,y) = 4x+6y-x^2-y^2$, $D = \{(x,y): 0 \le x \le 4, 0 \le y \le 5\}$.

Step 1 Find local extreme pts $f_x = 4-2x = 0$, $f_y = 6-2y = 0$

$x = 2$ $\quad y = 3$



critical pt: $(2,3)$

Step 2 Find extreme pt on the boundary

① when $x=0$, $0 \le y \le 5$, $f(x,y) = 6y - y^2 = f_1(y)$ $\quad f_1'(y) = 6-2y$, $y=3$

$(0,3)$, $(0,0)$, $(0,5)$ boundary pts

② when $x=4$, $0 \le y \le 5$, $f(x,y) = 16+6y-16-y^2 = 6y-y^2 = f_2(y)$ $\quad f_2'(y) = 6-2y$

$y = 3$

$(4,3)$, $(4,0)$, $(4,5)$

③ when $y=0$, $0 \le x \le 4$, $f(x,y) = 4x - x^2 = f_3(x)$ $\quad f_3'(x) = 4-2x \Rightarrow x=2$

$(2,0)$, $(0,0)$, $(4,0)$

④ when $y=5$; $0 \le x \le 4$, $f(x,y) = 4x+30-x^2-25 = 4x-x^2-25 = f_4(x)$ $\quad f_4'(x) = 4-2x$

$\Rightarrow x = 2$

$(2,5)$, $(0,5)$, $(4,5)$

Step 3

(x,y)	$(2,3)$	$(0,3)$	$(4,3)$	$(2,0)$	$(2,5)$	$(0,0)$	$(0,5)$	$(4,0)$	$(4,5)$
$f(x,y)$	17	9	9	4	9	0	5	0	5
	ab. max					ab. min		ab. min	

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(2) L_1: $x = 1+t$, $y = 1+6t$, $z = 2t$

closet to $(1, 2, 3)$ (this is not really a multivariable Calc problem!)

$$\text{Distance} = \sqrt{(1+t-1)^2 + (1+6t-2)^2 + (2t-3)^2} = \sqrt{t^2 + (6t-1)^2 + (2t-3)^2}$$

$$f(t) = t^2 + (6t-1)^2 + (2t-3)^2 = t^2 + 36t^2 - 12t + 1 + 4t^2 - 12t + 9$$

$$= 41t^2 - 24t + 10$$

$$f'(t) = 82t - 24 = 0 \Rightarrow t = \frac{24}{82} = \frac{12}{41}$$

$$f''(t) = 82 > 0$$

So when $t = \frac{12}{4}$, $f(t)$ achieves the minimum.

point $\left(\frac{12}{41} + 1,\ 1 + \frac{12}{41}\cdot 6,\ 2\cdot\frac{12}{41}\right) = \left(\frac{53}{41},\ \frac{113}{41},\ \frac{24}{41}\right)$

(3) L_1: $x = 1+t$, $y = 1+6t$, $z = 2t$

L_2: $x = 1+2s$, $y = 5+15s$, $z = -2+6s$ distance

$$\text{Distance} = \sqrt{(1+t-1-2s)^2 + (1+6t-5-15s)^2 + (2t+2-6s)^2}$$

$$= \sqrt{(t-2s)^2 + (6t-15s-4)^2 + (2t-6s+2)^2}$$

$$f(t,s) = (t-2s)^2 + (6t-15s-4)^2 + (2t-6s-2)^2$$

$$\cancel{= t^2 - 4st + 4s^2 +}$$

$$f_t = 2(t-2s) + 2(6t-15s-4)\cdot 6 + 2(2t-6s-2)\cdot 2 = 0$$

$$= 2\,(t - 2s + 36t - 90s - 24 + 4t - 12s - 4) = 0$$

$$= 2\,(41t - 104s - 28) = 0$$

$$f_s = 2(t-2s)(-2) + 2(6t-15s-4)(-15) + 2(2t-6s-2)(-6)$$

$$= -2\,(2t - 4s + 90t - 225s - 60 + 12t - 36s - 12) = -2\,(104t - 265s - 72) = 0$$

$$\begin{cases} f_t = 0 \\ f_s = 0 \end{cases} \Rightarrow \begin{cases} 41t - 104s - 28 = 0 \\ 104t - 265s - 72 = 0 \end{cases} \Rightarrow \begin{cases} t = -\frac{68}{49} \\ s = -\frac{40}{49} \end{cases}$$

$$H = \begin{pmatrix} f_{tt} & f_{st} \\ f_{ts} & f_{ss} \end{pmatrix} = 2\begin{pmatrix} 41 & -104 \\ -104 & 265 \end{pmatrix} \Rightarrow \text{local min.}$$

$$f\left(-\frac{68}{49}, -\frac{40}{49}\right) = \left(-\frac{68}{49} + \frac{80}{49}\right)^2 + \left(-\frac{6\cdot 68}{49} + \frac{15\cdot 40}{49} - 4\right)^2 + \left(\frac{2\cdot 68}{49} + \frac{6\cdot 40}{49} - 2\right)^2$$

$$= \frac{1}{49^2}(12)^2 + \frac{1}{49^2}(4)^2 + \frac{1}{49}(6)^2 = \frac{1}{49^2}196 = \left(\frac{14}{49}\right)^2 = \left(\frac{2}{7}\right)^2$$

distance $= \frac{2}{7}$.

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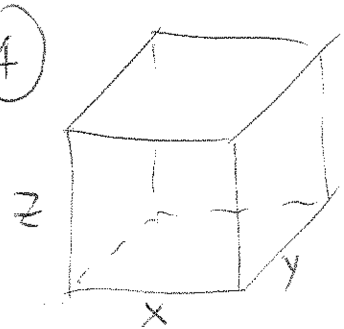

$V = xyz$.

$C = 5xy + 2xz + 2yz$

min C given $V = xyz$, $\Rightarrow z = \frac{V}{xy}$.

$$C(x,y) = 5xy + 2(x+y)z = 5xy + 2(x+y)\cdot\frac{V}{xy}$$

$$= 5xy + 2V\cdot\frac{x+y}{xy}, \quad (x,y) \in (\mathbb{R}_+)^2 = (0,\infty)\times(0,\infty)$$

$$= 5xy + \frac{2V}{y} + \frac{2V}{x}$$

$$C_x = 5y - \frac{2V}{x^2} = 0 \Rightarrow y = \frac{2V}{5x^2}$$

$$C_y = 5x - \frac{2V}{y^2} = 0$$

$$\Rightarrow 5x - \frac{2V}{\left(\frac{2V}{5x^2}\right)^2} = 5x - \frac{25x^4}{2V} = 0$$

$$10Vx - 25x^4 = 5x(2V - 5x^3) = 0$$

$x = 0$ (not valid) $\quad 2V - 5x^3 = 0 \Rightarrow x^3 = \frac{2V}{5} \Rightarrow x = \sqrt[3]{\frac{2V}{5}}$

$$y = \frac{2V}{5x^2} = \frac{2V}{5\left(\frac{2V}{5}\right)^{\frac{2}{3}}} = \sqrt[3]{\frac{2V}{5}} \qquad z = \frac{V}{xy} = \frac{V}{\left(\frac{2V}{5}\right)^{2/3}} = \sqrt[3]{\frac{2V}{5}} \cdot \frac{5}{2}$$

$$C_{xx} = \frac{4V}{x^3} \quad C_{xy} = 5 = C_{yx}, \quad C_{yy} = \frac{4V}{y^3}$$

$$H = \begin{pmatrix} 10 & 5 \\ 5 & 10 \end{pmatrix} \Rightarrow \text{min}. \qquad (x, y, z) = \sqrt[3]{\frac{2V}{5}}\left(1, 1, \frac{5}{2}\right)$$

(5) $x, y, z > 0$. $\quad x + y + z = 12$

$x^2 + y^2 + z^2$ smallest.

$$z = 12 - x - y \Rightarrow f(x, y) = x^2 + y^2 + (12 - x - y)^2, \quad (x, y) \in (0, \infty) \times (0, \infty)$$

$$f_x = 2x + 2(12 - x - y)(-1) = 0 \Rightarrow 2x - 24 + 2x + 2y = 0$$

$$4x + 2y = 24 \Rightarrow 2x + y = 12$$

$$f_y = 2y + 2(12 - x - y)(-1) = 0 \Rightarrow x + 2y = 12$$

$$\begin{cases} 2x + y = 12 \\ x + 2y = 12 \end{cases} \Rightarrow x = y = 4 \qquad H = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \Rightarrow \text{min}.$$

$(x, y, z) = (4, 4, 4)$ will make $x^2 + y^2 + z^2$ smallest.

(6) $f(x,y) = 2x^2 + 3y^2 - 4x - 5 + 6y$ on $D = \{(x,y) : x^2 + y^2 \le 16\}$

<u>Step 1</u> local min/max
$$f_x = 4x - 4 = 0 \Rightarrow x = 1$$
$$f_y = 6y + 6 = 0 \Rightarrow y = -1 \Rightarrow (1, -1)$$

<u>Step 2</u> boundary min/max $\quad \underset{max}{min}\ f(x,y) : x^2 + y^2 = 16 \quad ?$

(New method in next section)

but $\begin{cases} x = 4\cos\theta \\ y = 4\sin\theta \end{cases} \Rightarrow x^2 + y^2 = 16$

$$f(x,y) = 2 \cdot 16\cos^2\theta + 3 \cdot 16\sin^2\theta - 4 \cdot 4\cos\theta + 6 \cdot 4\sin\theta - 5$$

$$\Rightarrow g(\theta) = 32\cos^2\theta + 48\sin^2\theta - 16\cos\theta + 24\sin\theta - 5$$
$$= 32 + 16\sin^2\theta - 16\cos\theta + 24\sin\theta - 5$$
$$= 16\sin^2\theta - 16\cos\theta + 24\sin\theta + 27$$

$$g'(\theta) = 32\sin\theta\cos\theta + 16\sin\theta - 24\cos\theta = 0$$
$$= 8(4\sin\theta\cos\theta + 2\sin\theta - 3\cos\theta) \quad \text{not so easy}$$

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alternative way $\quad y = \pm\sqrt{16 - x^2}$

$$f(x,y) = 2x^2 + 3(16 - x^2) - 4x \pm 6\sqrt{16 - x^2} - 5$$
$$= -x^2 - 4x \pm 6\sqrt{16 - x^2} + 43 \qquad \text{not so easy}$$

$$-4 \le x \le 4$$