Functions of several variables

$z = f(x, y)$: two variables, one function value
Domain: a subset of $\mathbb{R}^2$, Range: a subset of $\mathbb{R}^1$
(Compare with vector-valued function: $\mathbf{r}(t) = \langle x(t), y(t) \rangle$
Domain: a subset of $\mathbb{R}^1$, Range: a subset of $\mathbb{R}^2$)

Examples:
(1) $z = \sqrt{x^2 + y^2}$ (distance of (x, y) to the origin)
(2) The temperature $T(x, y)$ of different locations of Virginia
(3) The elevation $E(x, y)$ of different locations of Virginia
(4) Cobb-Douglas production function $P(L, K) = aL^{\alpha}K^{1-\alpha}$ (L: labor, K: capital)

Four ways of expressing a function:
(a) verbally (a description in words); (b) algebraically (a formula)
(c) numerically (a spread sheet); (d) visually (a graph)

Graph of the function

$z = f(x, y)$: a 3-d graph
Linear function: $z = ax + by + c$ (graph is a plane in 3-d space)
(half of) sphere: $z = \sqrt{9 - x^2 - y^2}$
Elliptic paraboloid: $z = 2x^2 + y^2$

Level curves (contour map)
For $z = f(x, y)$, $f(x, y) = c$ is a level curve (a curve on xy-plane)
contour map is a collection of level curves

Examples:
(1) Constant elevation curves on topographic maps; (hiking map)
(2) Isothermal curves for temperature function. (whether forecast)

Example 1:
Let (a) $f(x, y) = \ln(25 - x^2 - y^2)$; (b) $f(x, y) = \sqrt{x^2 - y^2}$.

1. Evaluate $f(2, 1)$;
2. Find the domain and range of f;
3. Sketch the graph of f;
4. Sketch the level curves of f.

Limit and continuity

Review of Calculus I: $y = f(x)$,
Limit, $\lim_{x \to a} f(x) = L$: the value of $f(x, y)$ is very close to L as long as x is sufficiently close to a, $y = f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

<u>Which function is continuous?</u>
$f(x) = x^n$, e^x, $\ln x$, $\sin(x)$, $\cos(x)$ are continuous in their domains;
<u>Which function is not continuous?</u>
(1) $f(x) = 1/x$ at $x = 0$;
(2) $f(x) = 1$ for $x < 0$, $f(x) = 2$ for $x \geq 0$.

Multivariate function (Good cases): "combinations" of 1-d continuous functions are continuous

Example 2: The following functions are continuous in their domains
(1) $f(x, y) = 2xy - y^2 - sin(x)e^{x+y}$
(2) $f(x, y) = \ln(x^3 - e^y)$

Limit and continuity

1-d: $\lim_{x\to a} f(x) = L$ if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$ (limit from the left and the right)
2-d? limit $\lim_{(x,y)\to(a,b)} f(x,y) = L$ exists if the limit from any path approaching to (a,b)
limit does not exist if limits from two different paths are different

If $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, then the function $f(x,y)$ is continuous at the point (a,b).

Example 3: Determine whether the limit of $f(x,y)$ at $(0,0)$ exists

(a) $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$; (b) $f(x,y) = \dfrac{x^3y}{x^4+y^4}$;

(c) $f(x,y) = \dfrac{xy^3}{x^2+y^6}$; (d) $f(x,y) = \dfrac{xy^3}{x^2+y^2}$.

Example 4: Determine the set of points at which the function is continuous:
$f(x,y) = \dfrac{xy}{x^2+xy+y^2}$ if $(x,y) \neq (0,0)$, $f(x,y) = 0$ if $(x,y) = (0,0)$.