Projectile problem

A projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$. Assuming that air resistance is negligible and the only external force is due to gravity, find the position function $\mathbf{r}(t)$ of the projectile. What value of maximizes the range (the horizontal distance traveled)?

Solution $\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0$

If $\mathbf{v}_0 = v_0\cos(\alpha)\mathbf{i} + v_0\sin(\alpha)\mathbf{j}$, then $\mathbf{r}(t) = v_0\cos(\alpha)t\mathbf{i} + \left[v_0\sin(\alpha)t - \frac{1}{2}gt^2\right]\mathbf{j}$.

Velocity $\mathbf{v}(t) = v_0\cos(\alpha)\mathbf{i} + (v_0\sin(\alpha) - gt)\mathbf{j}$.

When the projectile reaches the highest point, the vertical velocity is zero so the time is $T_1 = v_0\sin(\alpha)/g$.

When the projectile hits the ground (impact time), the vertical position is 0 so $T_2 = 2v_0\sin(\alpha)/g = 2T_1$. And the range is the horizontal part of $\mathbf{r}(T_2) = 2v_0^2\sin(\alpha)\cos(\alpha)/g = (v_0^2/g)\sin(2\alpha)$. That is at maximum if $2\alpha = \pi/2$, or $\alpha = \pi/4$.

Example. (Page 894 problem 23) A projectile is fired with an initial speed of $200m/s$ and angle of elevation 60°. Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

Story of Tycho Brahe, Johannes Kepler and Issac Newton

  

Tycho Brahe (1546-1601), Johannes Kepler (1571-1630), Isaac Newton (1642-1727)

A paradigmatic account of the uses of mathematics in the natural sciences comes, in deliberately oversimplified fashion, from the classic sequence of Brahe, Kepler, Newton: observed facts, patterns that give coherence to the observations, fundamental laws that explain the patterns. These days, mathematics enters at every stage: in designing the experiment, in seeking the patterns, in reaching to understand underlying mechanisms.

——From "Uses and Abuses of Mathematics in Biology"
Sir Robert May, *Science*, February 6, 2004.

The following story is based on
http://csep10.phys.utk.edu/astr161/lect/index.html
http://csep10.phys.utk.edu/astr161/lect/history/brahe.html
http://csep10.phys.utk.edu/astr161/lect/history/kepler.html
http://csep10.phys.utk.edu/astr161/lect/history/newton.html

Tycho Brahe

A Danish nobleman, Tycho Brahe (1546-1601), made important contributions by devising the most precise instruments available before the invention of the telescope for observing the heavens. Brahe made his observations from Uraniborg, on an island in the sound between Denmark and Sweden called Hveen. The instruments of Brahe allowed him to determine more precisely than had been possible the detailed motions of the planets. In particular, Brahe compiled extensive data on the planet Mars, which would later prove crucial to Kepler in his formulation of the laws of planetary motion because it would be sufficiently precise to demonstrate that the orbit of Mars was not a circle but an ellipse.

Brahe proposed a model of the Solar System that was intermediate between the Ptolemaic and Copernican models (it had the Earth at the center). It proved to be incorrect, but was the most widely accepted model of the Solar System for a time. In the interplay between quantitative observation and theoretical construction that characterizes the development of modern science, we have seen that Brahe was the master of the first but was deficient in the second. The next great development in the history of astronomy was the theoretical intuition of Johannes Kepler (1571-1630), a German who went to Prague to become Brahe's assistant.

Kepler and Brahe did not get along well. Brahe apparently mistrusted Kepler, fearing that his bright young assistant might eclipse him as the premiere astronomer of his day. He therefore let Kepler see only part of his voluminous data.

Kepler: assistant to Brahe

He set Kepler the task of understanding the orbit of the planet Mars, which was particularly troublesome. It is believed that part of the motivation for giving the Mars problem to Kepler was that it was difficult, and Brahe hoped it would occupy Kepler while Brahe worked on his theory of the Solar System. In a supreme irony, it was precisely the Martian data that allowed Kepler to formulate the correct laws of planetary motion, thus eventually achieving a place in the development of astronomy far surpassing that of Brahe.

Unlike Brahe, Kepler believed firmly in the Copernican system. In retrospect, the reason that the orbit of Mars was particularly difficult was that Copernicus had correctly placed the Sun at the center of the Solar System, but had erred in assuming the orbits of the planets to be circles. It fell to Kepler to provide the final piece of the puzzle: after a long struggle, in which he tried mightily to avoid his eventual conclusion, Kepler was forced finally to the realization that the orbits of the planets were not the circles demanded by Aristotle and assumed implicitly by Copernicus, but were instead the "flattened circles" that geometers call ellipses.

The irony noted above lies in the realization that the difficulties with the Martian orbit derive precisely from the fact that the orbit of Mars was the most elliptical of the planets for which Brahe had extensive data. Thus Brahe had unwittingly given Kepler the very part of his data that would allow Kepler to eventually formulate the correct theory of the Solar System and thereby to banish Brahe's own theory!

The Laws of Planetary Motion

Kepler obtained Brahe's data after his death despite the attempts by Brahe's family to keep the data from him in the hope of monetary gain. There is some evidence that Kepler obtained the data by less than legal means; it is fortunate for the development of modern astronomy that he was successful. Utilizing the voluminous and precise data of Brahe, Kepler was eventually able to build on the realization that the orbits of the planets were ellipses to formulate his Three Laws of Planetary Motion:

I. The orbits of the planets are ellipses, with the Sun at one focus of the ellipse.
II. The line joining the planet to the Sun sweeps out equal areas in equal times as the planet travels around the ellipse.
III. The ratio of the squares of the revolutionary periods for two planets is equal to the ratio of the cubes of their semimajor axes.

Kepler had proposed three Laws of Planetary motion based on the systematics that he found in Brahe's data. These Laws were supposed to apply only to the motions of the planets; they said nothing about any other motion in the Universe. Further, they were purely empirical: they worked, but no one knew a fundamental reason WHY they should work.

Isaac Newton

Newton changed all of that. First, he demonstrated that the motion of objects on the Earth could be described by three new Laws of motion, and then he went on to show that Kepler's three Laws of Planetary Motion were but special cases of Newton's three Laws if a force of a particular kind (what we now know to be the gravitational force) were postulated to exist between all objects in the Universe having mass. In fact, Newton went even further: he showed that Kepler's Laws of planetary motion were only approximately correct, and supplied the quantitative corrections that with careful observations proved to be valid.

Kepler's laws and Newton's laws taken together imply that the force that holds the planets in their orbits by continuously changing the planet's velocity so that it follows an elliptical path is (1) directed toward the Sun from the planet, (2) is proportional to the product of masses for the Sun and planet, and (3) is inversely proportional to the square of the planet-Sun separation.

`http://csep10.phys.utk.edu/astr161/lect/history/newtonkepler.html`

Christopher Wren, Robert Hooke, Edmond Halley

http://faculty.wcas.northwestern.edu/~infocom/Ideas/newton.html

One day in 1684, three men had a conversation during a meeting of the Royal Philosophical Society in London. This may not sound like the kind of event which changes the world, but this was not your run-of-the-mill conversation. The three men were Sir Christopher Wren, the famous architect who had designed much of Oxford University and also 51 churches in London, including St. Paul's Cathedral; Robert Hooke, the noted physicist and astronomer who had discovered the force law for springs and detected the rotation of Jupiter; and Sir Edmond Halley, the prominent astronomer who first mapped the stars in the southern hemisphere and also predicted the return of the famous comet now named after him. The Royal Philosophical Society was actually a scientific society, and the men were not there to discuss theology. They had met to discuss the mysteries of the Universe.

By this time, there were only a few die-hards left (among educated men in Protestant England, at least) who still believed that everything moved around the Earth. Johannes Kepler's mathematical analyses of planetary observations, published between 1609 and 1629, had shown that only planets moving on elliptical orbits around a central Sun could adequately explain what astronomers observed – and the Earth was not excepted. It too had to move about the Sun. Galileo's 1642 publication of his Treatise on Two Sciences had demolished Aristotelian notions of "absolute" velocity, and shown how it was possible for the Earth to both rotate on its axis and race around the Sun, and yet not so much as ripple the tea in the cups of three gentlemen having a spirited discussion.

Christopher Wren, Robert Hooke, Edmond Halley

The conversation turned to the nature of the unknown force which many thought had to emanate from the Sun, to hold the planets in their paths as they orbited. Without some type of invisible force acting, the planets would break away, exactly like a revolving weight at the end of a string if the string breaks. Hook and Halley were of the opinion that the unknown force should be proportional to $1/r^2$.

Both Halley and Hook agreed that suppositions and analogies were no proof that an inverse-square force was radiating from the Sun. The real test would come when it was proven that you could (or could not) derive an inverse-square force law from Kepler's Laws of Planetary Motion, and vice versa. Kepler's Laws were based on direct astronomical observations, and therefore represented simple fact. If you could mathematically connect Kepler's Laws to an inverse-square force law with a rigorous proof, then the presence of an inverse-square force would also be irrefutably proved.

Robert Hooke, as it happened, was the kind of man whose great education and intelligence was surpassed only by his incredible ego and almost unbearable arrogance. He was famous for his public quarrels. (Halley and Wren, on the other hand, were equally famous for being gentlemen of the highest order.) Hooke concluded the conversation by boasting that he could produce the necessary proof within a couple of months.

Christopher Wren, Robert Hooke, Edmond Halley

Hooke had made this boast before. In fact, he had made this boast many times before. Possibly Wren had gotten tired of hearing it, or maybe he was intrigued when Halley said that maybe he would try his hand at producing a proof. Whatever his motivation, Wren – in a very sporting gesture offered either Hooke or Halley 40 shillings if they could produce a proof within two months. No sooner said than done, and Halley and Hooke accepted the challenge.

The two months passed. Halley admitted to Wren that he hadn't made a dent in the problem. Hooke well, as usual, he claimed he had gotten busy with some other things, and hadn't been able to give the problem much thought. Unbeknownst to both men, the problem they had set themselves cannot be solved without the use of calculus – which had not yet been developed. The problem was a very hard one.

Now, however, the problem had gotten underneath Halley's skin. He began thinking about who might be able to solve it, if he and Hooke could not. His thoughts turned to an obscure mathematician and physicist by the name of Isaac Newton, who was a Professor at Cambridge University. Newton was an odd eccentric, but he was astonishingly good at mathematics. Halley decided he would visit Newton and ask him about the inverse-square problem.

Newton

Halley made the trip to Cambridge, and asked Newton how he supposed the planets might move under the influence of an inverse-square law. To Halley's complete surprise, Newton immediately told him that he knew the planets would move in ellipses with the Sun at one focus, just as Kepler's Laws said, because he had calculated it years ago! (Newton hadn't told anybody, of course. And by the way, that old story that Newton had begun thinking about gravitation when he was sitting in a garden one day, and saw an apple fall from a tree? Surprisingly, it's completely true.)

The perspicacious among you may well wonder how Newton had been able to do his orbital calculations if calculus is required, but calculus hadn't been developed yet. Well, small problems of this sort are not insurmountable if you are as bright as Isaac Newton. He had invented an early form of calculus when he was 22, and he hadn't told anybody about that, either.

Halley wanted to see the written solution, naturally, but Newton told Halley that he couldn't find his notes. Many scholars now believe that Newton had only worked out the solution in a rough form, and was hesitant to let anyone see his work until he had polished it up some. One popular, but undoubtedly mythological, version of the Newton-Halley story has Newton's notes being accidentally burned in the fireplace after his dog, Diamond, overturns a table. Newton is supposed to have cried out, no doubt in a tragic voice, "Diamond! You little know what worlds you have destroyed¡'Ahem. In any case, Halley made Newton promise to send him a complete solution, and Newton said he would.
What followed next would change the world, because it created modern physics.

Newton

As Newton began work on completing his notes for Halley, he found himself being drawn deeper and deeper into the theory of motion and gravitation. Something possessed him. In a white-hot fury he began working on the theory almost around the clock, sometimes going without sleep for days at a time. He began having all his meals delivered to his study, but more often than not, Newton's man-servant would come in hours later and find the food right where he'd left it, completely untouched. The servant (who by remarkable coincidence was also named Newton, no relation to Isaac) wrote later that he was not always certain if Newton was even aware of his coming and going. After about three months, Newton finally sent the promised notes to Halley – but he did not stop working.

Halley was astonished when he received the notes. Instead of being just a solution to the specific inverse-square problem he had discussed with Newton, the notes were a sweeping treatise on force, motion, gravitation, inertia, and the orbits of the planets. In them, Newton presented his Law of Universal Gravitation, and also systematically laid out the rules governing all moving bodies. These rules are known today as Newton's Laws of Motion (even though some of the ideas had been anticipated previously by Galileo and Descartes).

In the papers he sent to Halley, Newton proved that the planets must obey Kepler's Laws if universal gravitation is correct. This alone would have been a great achievement. But Newton had gone far beyond that, and created a new paradigm of physics. Halley immediately realized the immense importance of the papers, and he urged Newton to publish a book.

Newton

Halley realized that he was reading manuscript pages from what was quite probably the most important scientific treatise ever written and Newton didn't want to publish it! Halley redoubled his urging. He offered to personally take care of all the details of the publishing process for Newton, and to pay all the printing costs. (Which he eventually did. The world is much in debt to Edmond Halley.)

And then Halley received some unlikely help, from the ever-healthy ego of Robert Hooke. Hooke had learned of Newton's work from Halley and others (Newton's work was unpublished, but word of what he was working on had gotten around), and Hooke had begun claiming that there were errors in it. And then he began claiming that the whole idea of an inverse-square law was his, and that Newton had stolen it from him. This outraged Newton, who was not at all the kind to forgive and forget. He was still smoldering with anger over Hooke's criticism, 15 years ago, of his papers on light and color. Newton decided that the best way to silence Hooke was to publish, so that everyone could see how far advanced his work was. Between the prodding from Halley and the boasting from Hooke, Newton broke his accustomed reclusiveness and in 1687 published the book we today call the **Principia**.

But after two years of nearly nonstop labor, the Principia had burned Newton out. He never worked on mathematics or physics again. In one of the more startling career changes in physics history, Newton resigned his Professorship at Cambridge and went to London, where he proceeded to skillfully parlay his newly-found fame into a high-paying position in the Royal Mint in 1696, appointed by King William III and Queen Mary II.

Newton's laws

Second law of motion: $\mathbf{F} = m\mathbf{a}$, Law of gravitation: $\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^3}\mathbf{r}$

M: mass of sun, m: mass of planet, G: gravitational constant

So $\ddot{\mathbf{r}} = \mathbf{a} = -\frac{GM}{|\mathbf{r}|^3}\mathbf{r}$

Kepler's First Law: The orbits of the planets are ellipses, with the Sun at one focus of the ellipse.

$R = \frac{ed}{1 + e\cos\theta}$ (in polar coordinate)