Vector-valued function

Vector-valued function:
Vector equation: $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$
Parametric equation: $x = f(t)$, $y = g(t)$, $z = h(t)$

Simple rules:
Limit: $\lim_{t \to a} \mathbf{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$
Derivative: $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$
Integral: $\int_a^b \mathbf{r}(t)dt = \langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \rangle$

What is a Vector-valued function? (in the 3-d graph)
It is a space curve! parametric equation with t as a parameter

Special space curves:
Lines: $\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$
Helix: $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

In 1953, James Watson and Francis Crick found that the structure of the DNA molecule is that of two linked, parallel helices that are interwined.

Derivatives and physics

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$.

Geometry: $\mathbf{r}'(t_0)$ is a tangent vector to the curve $\mathbf{r}(t)$ at $t = t_0$ (if $\mathbf{r}'(t_0) \neq 0$), and the line through $\mathbf{r}(t_0)$ with direction $\mathbf{r}'(t_0)$ is the tangent line of $\mathbf{r}(t)$ at $t = t_0$.

tangent line: $\mathbf{u}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$,
unit tangent vector: $\mathbf{T}(t_0) = \dfrac{\mathbf{r}'(t_0)}{|\mathbf{r}'(t_0)|}$

Physics:
$\mathbf{v}(t_0) = \mathbf{r}'(t_0)$ is the velocity of $\mathbf{r}(t)$ at $t = t_0$,
$|\mathbf{v}(t_0)|$ is the speed of $\mathbf{r}(t)$ at $t = t_0$,
and $\mathbf{a}(t_0) = \mathbf{v}'(t_0) = \mathbf{r}''(t_0)$ is the acceleration of $\mathbf{r}(t)$ at $t = t_0$.

Examples

1. Find the domain of the function $\mathbf{r}(t) = \langle \ln(4 - t^2), \sqrt{t+1}, \cos(2t) \rangle$.
2. Find the vector and parametric equation of the line segment that joins $P(-2, 4, 0)$ and $Q(6, -1, 2)$.
3. Find a vector function that represents the curve of intersection of the paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$.
4. Find the parametric equation of the tangent line of $\mathbf{r}(t) = \langle \ln t, 2\sqrt{t+1}, t^2 \rangle$ at $P(0, 2\sqrt{2}, 1)$; Find the unit tangent vector at P.
5. Find the velocity, speed, and acceleration of $\mathbf{r}(t) = \langle \ln(4 - t^2), \sqrt{t+1}, \cos(2t) \rangle$
6. Find the position function of the particle with acceleration $\mathbf{a}(t) = \langle t, t^2, \cos(2t) \rangle$, $\mathbf{v}(0) = \langle 1, 0, 1 \rangle$ and $\mathbf{r}(0) = \langle 0, 1, 0 \rangle$.

Arc-length

Arc-length of plane curve: (Calculus II)

$y = f(x)$, from (x_1, y_1) to (x_2, y_2): $\int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2}\,dx$

or $(x(t), y(t))$, $a \le t \le b$: $\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$

Arc-length of space curve:

$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt \text{ or } L = \int_a^b |\mathbf{r}'(t)|\,dt$$

(distance travelled equals to the integral of speed (not velocity))

Example:
(a) Find the arc-length of $\mathbf{r}(t) = \langle 2 \sin t, 5t, 2 \cos t \rangle$, $-10 \le t \le 10$.
(b) Find the length (correct to four decimal places) of the curve of intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.