Equation of a line in 2-d

1. $y = k(x - x_0) + y_0$ (coordinate equation: point-slope) k: slope,
2. $x = At + x_0$, $y = Bt + y_0$, (parametric equation)
   $\langle A, B \rangle$: direction of line
3. $\langle a, b \rangle \cdot \langle x - x_0, y - y_0 \rangle = 0$,
   (vector equation) $\langle a, b \rangle$: normal direction
   (meaning: all points such that $\langle x - x_0, y - y_0 \rangle$ is orthogonal to $\langle a, b \rangle$)
4. $ax + by = c$ (a general form)

Example 1: Find the three forms of the equations of the line passing thorough $(1, 2)$ and $(-3, 7)$.

Equation of a plane in 3-d

vector equation: $\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$
$\langle a, b, c\rangle$: normal vector, (x_0, y_0, z_0) a point on the plane
(one point and one direction can determine a plane)

a general form: $ax + by + cz = d$ ($\langle a, b, c\rangle$: normal vector)

Example 2:

1. Find the equation of the plane that passes through the points $P(1, 2, 3)$, $Q(2, -4, 9)$ and $R(0, 5, -4)$.
2. Find the angle between the planes $x - y + z = 4$ and $3x + 4y - 5z = 0$.
3. Find the equation of the plane through the point $P(1, 2, 3)$ and parallel to $2x + 4y + 7z = 18$

Equation of a line in 3-d

Parametric equation: $x = At + x_0$, $y = Bt + y_0$, $z = Ct + z_0$
$\langle A, B, C \rangle$: direction of line

Vector version: $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$, where $\mathbf{r_0}$ is a vector from origin to one of the points on the line, and $\mathbf{v}$ is the direction of the line.

Symmetric equation: $t = \dfrac{x - x_0}{A} = \dfrac{y - y_0}{B} = \dfrac{z - z_0}{C}$
(IF $A \neq 0$, $B \neq 0$, $C \neq 0$)

Example 3:

1. Find the parametric and symmetric equations of the line passing through $P(1, 2, 3)$ and $Q(-4, 7, 0)$.
2. Find the parametric and symmetric equations of the line of z-axis.
3. Find the equation of the line where the planes $x + y + z = 0$ and $2x - 3y + 7z = 9$ intersect.

Summary:
Equation of plane: $\overrightarrow{n} \cdot \overrightarrow{r - r_0} = 0$, where $\overrightarrow{n}$ is the normal vector, r_0 is a point on the plane.
Equation of the line: $\overrightarrow{r - r_0} = t\overrightarrow{n}$, where r_0 is a point on the line and $\overrightarrow{n}$ is the direction of the line.

The relations of lines and planes

In 2-d, two lines are either parallel or intersecting
In 3-d, two planes are either parallel or intersecting
(parallel if their normal vectors are parallel)
In 3-d, two lines can be parallel, intersecting, or neither (skew)

Example 4:

1. Determine the relation of planes $x + 2y + 3z = 10$ and $3x + 6y + 9z = 0$
2. Determine the relation of lines $x = 2t + 3,\ y = -3t - 4,\ z = 5t + 7$ and $x = 4s - 9,\ y = 8s - 3,\ z = -3s + 5$.
3. Determine the relation of lines $x = 2t + 3,\ y = -3t - 4,\ z = 5t + 7$ and $x = 4s + 3,\ y = -6s - 4,\ z = 10s + 7$.

Distances

The distance from a point $P(x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$:
$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 5:

1. Find the distance from $P(1, 2, 3)$ to the plane $5x - 7y + z = 22$.
2. Find the distance between two parallel planes $5x - 7y + z = 22$ and $5x - 7y + z = 1$.
3. Find the distance between the lines $x = 2t + 3$, $y = -3t - 4$, $z = 5t + 7$ and $x = 4s - 9$, $y = 8s - 3$, $z = -3s + 5$.