Determinant

Determinant: (more in linear algebra)
2-dimensional space or 2×2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \textbf{ Example}: \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$$

3-dimensional space or 3×3 matrix

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Example:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$
$$= 1 \cdot (-3) - 2 \cdot (-6) + 3 \cdot (-3) = 0.$$

$$\begin{vmatrix} 0 & -3 & 3 \\ 4 & -7 & 6 \\ 2 & 8 & -5 \end{vmatrix} = ?$$

Cross product

Cross product: (only for 3-d !!)

If $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, and $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

Properties (from calculations)
(0) $\mathbf{u} \times \mathbf{v}$ is a vector! (compare: $\mathbf{u} \cdot \mathbf{v}$ is a scalar)
(1) $\mathbf{u} \times \mathbf{u} = 0$ (Compare: $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$)
(2) $\mathbf{u} \times \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.
(Compare: $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal)
(3) $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Right-hand rule: if you curl your right hand around from $\mathbf{u}$ to $\mathbf{v}$, then your thumb points $\mathbf{u} \times \mathbf{v}$. (similar to Section 12.1.)

More rules

More interesting rules:
(4) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ (Compare: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$)
(5) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
(6) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$
(7) $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
(8) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

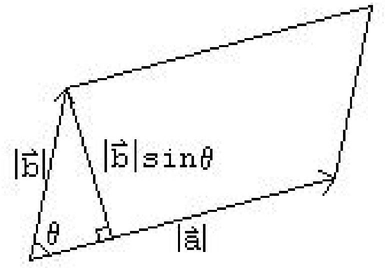
Not so interesting rules:
(9) $(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v})$,
(10) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Why cross product is useful?

Cross product and geometry (1)

$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ ($0 \leq \theta \leq \pi$).



One use of cross product:
The area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is $|\mathbf{u} \times \mathbf{v}|$
and the area of the triangle determined by $\mathbf{u}$ and $\mathbf{v}$ is $\frac{1}{2}|\mathbf{u} \times \mathbf{v}|$

Cross product and geometry (2)

$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called **scalar triple product**

Another use of cross product:
The volume of the parallelepiped determined by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is
$(\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, and $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k})$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

One trick:
(1)Three point P, Q and R are on the same line if $\overrightarrow{PQ} \times \overrightarrow{PR} = 0$
(2)Three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are on the same plane if $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$

Summary:

(scalar, dot, cross, triple) products

$\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

scalar product:
$c \cdot \mathbf{u} = c \cdot a_1\mathbf{i} + c \cdot a_2\mathbf{j} + c \cdot a_3\mathbf{k}$

dot product:
$\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos\theta$

cross product:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \sin\theta$ (area of parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$)

triple product:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(volume of the parallelepiped determined by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$)

Examples

1. Calculate $\mathbf{u} \times \mathbf{v}$, if $\mathbf{u} = \langle 5, 1, 4 \rangle$, and $\mathbf{v} = \langle -1, 0, 2 \rangle$; verify that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$;
2. Calculate $\mathbf{u} \times \mathbf{v}$, if $\mathbf{u} = \sin t\mathbf{i} - e^t\mathbf{j} + 30\mathbf{k}$, $\mathbf{v} = -\cos t\mathbf{i} + t^2\mathbf{j} - 7\mathbf{k}$.
3. Find the area of the triangle with vertices $P(2, 1, 5)$, $Q(-1, 3, 4)$ and $R(3, 0, 6)$; and find a vector orthogonal to the plane containing P, Q and R.
4. Find the volume of the parallelepiped determined by $\mathbf{u} = \langle 5, 1, 4 \rangle$, $\mathbf{v} = \langle -1, 0, 2 \rangle$, and $\mathbf{w} = \langle 3, -7, 5 \rangle$.
5. Prove $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$