Dot product

Given vector $\mathbf{u} = \langle a_1, b_1, c_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2, c_2 \rangle$, then the dot product is the number:

$$\mathbf{u} \cdot \mathbf{v} = a_1 a_2 + b_1 b_2 + c_1 c_2$$

Sometimes it is also called scalar product or inner product.

Example 1: Compute (a) $\langle 1, 1 \rangle \cdot \langle -1, 1 \rangle$, (b) $\langle 2, 3 \rangle \cdot \langle 2, 3 \rangle$, (c) $(2i - 7j - 5k) \cdot (-2i + 8j + 7k)$.

Observations:

- $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, $0 \cdot \mathbf{u} = 0$, $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$,
- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$, $\mathbf{u} \cdot \mathbf{v} = 0$ if $\mathbf{u}$ are $\mathbf{v}$ are perpendicular.

What is dot product, really?
Law of Cosine

\[ \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cos \theta, \]

where \( \theta \) is the angle between the vectors \( \mathbf{u} \) and \( \mathbf{v} \). The angle is between 0 and \( \pi \). Thus the angle between the vectors \( \mathbf{u} \) and \( \mathbf{v} \) can be calculated as

\[ \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|}. \]

In particular,
(a) \( \mathbf{u} \cdot \mathbf{v} = 0 \) if the angle is \( \pi/2 \) (perpendicular);
(b) \( \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \) if the angle is 0 (parallel);
(c) \( \mathbf{u} \cdot \mathbf{v} = -|\mathbf{u}| \cdot |\mathbf{v}| \) if the angle is \( \pi \) (opposite).
**Directional angles and Directional angles**

**Directional angles**: angles ($\alpha$, $\beta$ and $\gamma$) between the vector and the positive $x$-, $y$-, and $z$-axes.

**Directional Cosines**:

\[
\cos \alpha = \frac{\mathbf{u} \cdot \mathbf{i}}{||\mathbf{u}|| \cdot ||\mathbf{i}||}, \quad \cos \beta = \frac{\mathbf{u} \cdot \mathbf{j}}{||\mathbf{u}|| \cdot ||\mathbf{j}||}, \quad \cos \gamma = \frac{\mathbf{u} \cdot \mathbf{k}}{||\mathbf{u}|| \cdot ||\mathbf{k}||}.
\]

**Example 2**:
(a) Find the angle between the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 4, 0, -1 \rangle$.
(b) Determine whether the vectors are parallel, orthogonal, or neither: $\mathbf{u} = \langle 4, 6 \rangle$ and $\mathbf{v} = \langle -3, 2 \rangle$.
(c) Find the directional cosines and directional angles of the vector $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. 
Projections

The components of \( \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} \) are \( b_1 \mathbf{i}, b_2 \mathbf{j} \) and \( b_3 \mathbf{k} \); these components are also projections of \( \mathbf{b} \) onto \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \), respectively. And we can project a vector \( \mathbf{b} \) onto any vector \( \mathbf{a} \):

![Diagram of vector projection]

The length of the projection is \( |\mathbf{b}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \), and the direction of the projection is \( \frac{\mathbf{a}}{|\mathbf{a}|} \). Thus the projection is \( \text{proj}_\mathbf{a} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \cdot \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} \). (In the book, \( \text{comp}_\mathbf{a} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \) is called scalar projection, and \( \text{proj}_\mathbf{a} \mathbf{b} \) is called vector projection.)

**Example 3:** Find the scalar and vector projection of \( \mathbf{v} = \langle -1, -2, 2 \rangle \) onto \( \mathbf{w} = \langle 3, 3, 4 \rangle \).
The work

The work: The work done by a constant force $F$ from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is (work is a scalar)

$$W = F \cdot \overrightarrow{PQ}$$

Example 4: A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 50 N. the handle of the wagon is held at an angle of $30^\circ$ above the horizontal. How much work is done?