Dot product

Dot product Given vector $\mathbf{u} = \langle a_1, b_1, c_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2, c_2 \rangle$, then the dot product is the <u>number</u>:

$$\mathbf{u} \cdot \mathbf{v} = a_1 a_2 + b_1 b_2 + c_1 c_2$$

Sometime it is also called **scalar product** or **inner product**.

Example 1: Compute (a) $\langle 1, 1 \rangle \cdot \langle -1, 1 \rangle$, (b) $\langle 2, 3 \rangle \cdot \langle 2, 3 \rangle$, (c) $(2\mathbf{i} - 7\mathbf{j} - 5\mathbf{k}) \cdot (-2\mathbf{i} + 8\mathbf{j} + 7\mathbf{k})$.

Observations:
$\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, $\mathbf{0} \cdot \mathbf{u} = \mathbf{0}$, $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$,
$(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$ if $\mathbf{u}$ are $\mathbf{v}$ are perpendicular.

What is dot product, really?

Law of Cosine

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cos \theta,$$

where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. The angle is between 0 and π. Thus the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ can be calculated as

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|}.$$

In particular,
(a) $\mathbf{u} \cdot \mathbf{v} = 0$ if the angle is $\pi/2$ (perpendicular);
(b) $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}|$ if the angle is 0 (parallel);
(c) $\mathbf{u} \cdot \mathbf{v} = -|\mathbf{u}| \cdot |\mathbf{v}|$ if the angle is π (opposite).

Directional angles and Directional angles

Directional angles: angles (α, β and γ) between the vector and the positive x-, y-, and z-axes.

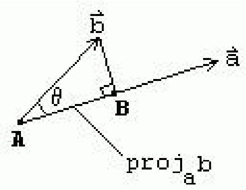
Directional Cosines:

$$\cos\alpha = \frac{\mathbf{u}\cdot\mathbf{i}}{|\mathbf{u}|\cdot|\mathbf{i}|}, \quad \cos\beta = \frac{\mathbf{u}\cdot\mathbf{j}}{|\mathbf{u}|\cdot|\mathbf{j}|}, \quad \cos\gamma = \frac{\mathbf{u}\cdot\mathbf{k}}{|\mathbf{u}|\cdot|\mathbf{k}|}.$$

Example 2:
(a) Find the angle between the vectors $\mathbf{u} = \langle 1, 2, 3\rangle$ and $\mathbf{v} = \langle 4, 0, -1\rangle$.
(b) Determine whether the vectors are parallel, orthogonal, or neither: $\mathbf{u} = \langle 4, 6\rangle$ and $\mathbf{v} = \langle -3, 2\rangle$.
(c) Find the directional cosines and directional angles of the vector $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Projections

The components of $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are $b_1\mathbf{i}$, $b_2\mathbf{j}$ and $b_3\mathbf{k}$; these components are also projections of $\mathbf{b}$ onto $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively. And we can project a vector $\mathbf{b}$ onto any vector $\mathbf{a}$:



The length of the projection is $|\mathbf{b}|\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$, and the direction of the of the projection is $\dfrac{\mathbf{a}}{|\mathbf{a}|}$. Thus the projection is $proj_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\cdot\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$. (In the book, $comp_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$ is called scalar projection, and $proj_{\mathbf{a}}\mathbf{b}$ is called vector projection.)

Example 3: Find the scalar and vector projection of $\mathbf{v} = \langle -1, -2, 2\rangle$ onto $\mathbf{w} = \langle 3, 3, 4\rangle$.

The work

The work: The work done by a constant force $\mathbf{F}$ from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is (work is a scalar)

$$W = \mathbf{F} \cdot \overrightarrow{PQ}$$

Example 4: A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 50 N. the handle of the wagon is held at an angle of 30° above the horizontal. How much work is done?