Change of variables

$x = r\cos\theta,\ y = r\sin\theta$ is a special change of variables in 2-D.

General change of variables: $x = g(u, v),\ y = h(u, v)$

This change is called a transformation of the domains, and usually differentiable (means h and g are differentiable functions). It is also one-to-one and onto (or bijective), which means one point (x, y) can only be the image of one (u, v) point.

Example 12: Find the region in xy plane which corresponds to the region in the following coordinate system.

1. $x = r\cos\theta,\ y = r\sin\theta,\ S_1 = \{(r,\theta) : r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2\}$
2. $x = v,\ y = u(1 + v^2),\ S_2 = \{(u, v) : 0 \le u \le 1, 0 \le v \le 1\}$.

Integral under change of variable:

1-D: $\displaystyle\int_{t=t_1}^{t=t_2} f(u(t))u'(t)dt = \int_{u_1}^{u_2} f(u)du,\ u_1 = u(t_1),\ u_2 = u(t_2)$

2-D: $\displaystyle\iint_R f(x, y)dxdy = \iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv$

R is the (x, y) region, S is the corresponding (u, v) region.

$T : (u, v) \to (x(u, v), y(u, v))$ is one-to-one and onto (bijective), and $\left|\dfrac{\partial(x, y)}{\partial(u, v)}\right|$ is the Jacobian of T

Jacobian

$dA = |\mathbf{dx}| \cdot |\mathbf{dy}| = |\mathbf{dx} \times \mathbf{dy}|$, $\mathbf{dx} = x_u(u,v)\mathbf{du} + x_v(u,v)\mathbf{dv}$, and
$\mathbf{dy} = y_u(u,v)\mathbf{du} + y_v(u,v)\mathbf{dv}$, so
$dA = |(x_u y_v - x_v y_u)\mathbf{du} \times \mathbf{dv}| = |x_u y_v - x_v y_u| \cdot |\mathbf{du}| \cdot |\mathbf{dv}|$

Jacobian of $T : (u,v) \to (x(u,v), y(u,v))$ is $x_u y_v - x_v y_u$, which is the determinant of the matrix:
$J = \begin{pmatrix} x_u(u,v) & x_v(u,v) \\ y_u(u,v) & y_v(u,v) \end{pmatrix}$, the Jacobian matrix

Example 13:

1. $\iint_R (4x + 8y)dA$, where R is the parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, and $(1,5)$: $x = (u+v)/4$, $y = (v-3u)/4$.
2. $\iint_R xydA$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$: $x = u/v$, $y = v$.

General 3-D change of variables

$x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$

Jacobian $\dfrac{\partial(x, y, z)}{\partial(u, v, w)} = det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$

$$\iiint_R f(x, y, z)dxdydz = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudv$$

Jacobian for cylindrical coordinates: $\dfrac{\partial(x, y, z)}{\partial(u, v, w)} = r$

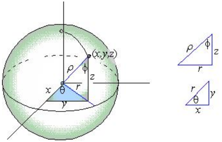
Spherical Coordinates

there are two angular directions now!

$x = \rho \sin(\phi) \cos(\theta)$, $y = \rho \sin(\phi) \sin(\theta)$, $z = \rho \cos(\phi)$
$r = \sqrt{x^2 + y^2} = \rho \sin(\phi)$, $z = \rho \cos(\phi)$
(θ is the angle in xy plane, ϕ is the angle between positive z-axis and vector $\langle x, y, z \rangle$).

Jacobian for cylindrical coordinates: $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \rho^2 \sin(\phi)$

$\rho = a$ (a sphere with radius a)
parametrization of a ball: $\{(\rho, \theta, \phi) : 0 \le \rho \le a, 0 \le \theta < 2\pi, 0 \le \phi < \pi\}$.



Latitude-longitude and spherical coordinates: center of earth is $(0,0,0)$, north pole is on positive z-axis, and positive x-axis is where the prime meridian (through Greenwich, England) intersects the equator. If the spherical coordinate is (ρ, θ, ϕ), then the latitude of P is $\alpha = 90^\circ - \phi^\circ$ and the longitude is $\beta = 360^\circ - \theta^\circ$. (this is true for northern hemisphere.)

Examples

Example 14

1. Find the volume of the ball with radius R.
2. An ice cream cone is approximately the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. Find its volume.
3. $\iiint_E xdV$, where E is the solid enclosed by the planes $z = 0$ and $z = x + y + 5$ and by the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.
4. Evaluate $\iiint_E e^{\sqrt{x^2+y^2+z^2}} dV$, where E is enclosed by the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.
5. $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) dzdxdy$

Smart Euler

Leonhard Euler (1707-1783)

1. Prove that $\int_0^1 \int_0^1 \frac{1}{1-xy} dxdy = \sum_{n=1}^{\infty} \frac{1}{n^2}$.
2. Solve the integral $\int_0^1 \int_0^1 \frac{1}{1-xy} dxdy$ by using the change of variables $x = \frac{u-v}{\sqrt{2}}$ and $y = \frac{u+v}{\sqrt{2}}$.

Leonhard Euler (1736): $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.