Triple integrals in cylindrical coordinate

Cylindrical coordinates: generalization of polar coordinates to 3-D
(so it is essentially 2-D change of variables)

$x = r\cos\theta,\ y = r\cos\theta,\ z = z$ (no change in z!)
$r^2 = x^2 + y^2,\ \tan\theta = \dfrac{y}{x},\ z = z.$

Equations:
$r = r_0$ (a cylinder, not a circle)
$\theta = \theta_0$ (a half-plane or a plane)
$r = f(\theta)$ (a surface generated by $r = f(\theta)$ in 2-D)

When to use cylindrical coordinates: circular regions except the 3-d ball

Integral in cylindrical coordinate:
$D = \{(x, y, z) : (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$
$= \{(r, \theta, z) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta),\ u_1(r\cos\theta, r\sin\theta) \le z \le$
$u_2(r\cos\theta, r\sin\theta)\}$

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz\right] dA$$
$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta$$

Examples

Example 11

1. Change from rectangular to cylindrical coordinates: $(x, y, z) = (2\sqrt{3}, 2, -1)$
2. Write the equations in cylindrical coordinates: (a) $3x + 3y + z = 6$, (b) $-x^2 - y^2 + z^2 = 1$.
3. Evaluate $\iiint_E x dV$ where E is enclosed by the planes $z = 0$ and $z = x + y + 5$ and by the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.
4. Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.
5. Evaluate the integral by changing to cylindrical coordinates:

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} \sqrt{x^2 + y^2} dzdydx$$