Triple integral

Domain: $B = [a, b] \times [c, d] \times [r, s]$
Function: $w = f(x, y, z)$ defined on B

Definition: $\Delta x = (b-a)/n$, $\Delta y = (d-c)/m$, $\Delta z = (s-r)/l$
$x_0 = a$, $x_1 = x_0 + \Delta x$, $\cdots$, $x_{i+1} = x_i + \Delta x$, $\cdots$, $x_n = b$;
$y_0 = c$, $y_1 = y_0 + \Delta y$, $\cdots$, $y_{j+1} = y_j + \Delta y$, $\cdots$, $y_m = d$;
$z_0 = r$, $z_1 = z_0 + \Delta z$, $\cdots$, $z_{k+1} = z_k + \Delta z$, $\cdots$, $z_l = s$;
$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ (a small box)
Pick a point $(x_{ijk}, y_{ijk}, z_{ijk})$ in B_{ijk}

Triple Riemann sum: $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} f(x_{ijk}, y_{ijk}, z_{ijk})\Delta V$

Triple integral on B: $\displaystyle\iiint_B f(x, y, z)\,dV = \lim_{m,n,l\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} f(x_{ijk}, y_{ijk}, z_{ijk})\Delta V$

Definition on a general domain (not a box): similar to 2-D

How to solve triple integral

Basically same idea as double integral, but difficulty is higher due to complex geometric structure

Rectangular box: **Fubini's Theorem**: If $f(x, y, z)$ is continuous on the rectangle $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)dV = \int_r^s \left[\int_c^d \left[\int_a^b f(x, y, z)dx\right] dy\right] dz$$

General 3-d region: (maybe not in that order)

$$\iiint_E f(x, y, z)dV = \int_r^s \left[\int_{g_1(z)}^{g_2(z)} \left[\int_{h_1(y,z)}^{h_2(y,z)} f(x, y, z)dx\right] dy\right] dz$$

Difficulty: need really good visualization of 3-D objects.

The volume of a solid region $E \in \mathbb{R}^3$ can be formulated as $\iiint_E 1dV$

The area of a solid region $S \in \mathbb{R}^2$ can be formulated as $\iint_S 1dA$

Examples

Example 9

1. $\int_0^1 \int_0^3 \int_0^{\sqrt{1-z^2}} ze^y \, dxdydz$
2. $\iiint_E xzdV$, where E is the tetrahedron with vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,1,1)$.
3. $\iiint_E xydV$, where E is bounded by the parabolic cylinders $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = x + y$.
4. Find the volume of the solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$.
5. $\iiint_B (z^3 + \sin y + 3)dV$ where B is the unit ball $x^2 + y^2 + z^2 \leq 1$.

Convert integrals into another form

For a double integral in a region $D \subset \mathbb{R}^2$, it can be evaluated by using an iterated integral in $dxdy$, or $dydx$, or using polar coordinate. And a conversion can be made between 3 forms.
For a triple integral in a region $D \subset \mathbb{R}^3$, it can be evaluated by using an iterated integral in $dxdydz$, or $dydxdz$, or $dxdzdy$, or $dzdxdy$, or $dydxdz$, or $dzdydx$, or using cylindrical coordinate, or using spherical coordinate. And a conversion can be made between 8 forms.

Example 10

1. Sketch the region of integration and change the order of integration:
$$\int_0^2 \int_{x^2}^4 f(x,y)dydx$$
2. Evaluate the iterated integral by converting to polar coordinate:
$$\int_0^a \int_{-\sqrt{a^2-y^2}}^0 x^2ydxdy$$
3. Write five other iterated integrals that are equal to $\int_0^1 \int_y^1 \int_0^z f(x,y,z)dxdzdy$