Disks and circles

Most natural regions: Rectangle and circular disks.

$\iint_D e^{x^2+y^2}\, dA$, where $D = \{(x, y) : x^2 + y^2 \leq 4\}$
Iterated integral (Fubini's Theorem) is about rectangular domains, and it is not natural for circular domains.

(x, y) is just one of the possible coordinate systems, (usually called Cartesian coordinate system). Next we will learn polar coordinate system in 2-D, and cylindrical coordinate system and spherical coordinate system in 3-D.

Polar coordinate system:
Introduction to polar coordinate system: page 678 (Section 10.3)
A point is represented by (r, θ), where $r \geq 0$, $-\infty < \theta < \infty$. r is the distance from the point to the origin, and θ is the angle between the vector from origin to the point and the positive x-axis. ($-r$ is also allowed sometime, $(-r, \theta) = (r, \theta + \pi)$.)

The origin in polar coordinate system is called pole.
Usually we will only use $r \geq 0$ and $0 \leq \theta < 2\pi$ as domain of polar coordinate system, and $(r, \theta + 2k\pi) = (r, \theta)$

Conversion between Polar and Cartesian coordinate systems:
$x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$ (or $\tan\theta = y/x$)

Functions in polar coordinate systems

1. $r = 3$ (a circle), (more general: $r = r_0$)
2. $\theta = \pi/2$ (a ray or a line) (more general: $\theta = \theta_0$)
3. $r = \theta$ (a spiral)
4. $r = 2\cos\theta$ (also a circle, convert to Cartesian coordinate)
5. $r = 1 + \sin\theta$ (cardioid)
6. $r = \cos(2\theta)$ (four-leaved rose) (more general: $r = \cos(n\theta)$, n integer)
7. $r = \cos(8\theta/5)$ (Try it on Matlab!) (more general: $r = \cos(n\theta)$, n rational number)
8. $r = \cos(\sqrt{2}\theta)$ (Chaos, Try it on Matlab!) (more general: $r = \cos(n\theta)$, n irrational number)

Integrals in polar coordinates

Example 7:
$\iint_R (3x+4y)dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Region in $x-y$: $\{(x,y): y \geq 0, 1 \leq x^2+y^2 \leq 4\}$, or
Region in $r-\theta$: $\{(r,\theta): 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

Key question: what is dA in polar coordinate?

$dA = |\mathbf{dx}| \cdot |\mathbf{dy}| = |\mathbf{dx} \times \mathbf{dy}|$, $\mathbf{dx} = \cos\theta \mathbf{dr} - r\sin\theta \mathbf{d}\theta$, and $\mathbf{dy} = \sin\theta \mathbf{dr} + r\cos\theta \mathbf{d}\theta$, so
$dA = |r\mathbf{dr} \times \mathbf{d}\theta| = r|\mathbf{dr}| \cdot |\mathbf{d}\theta|$

Convert integral to polar coordinate:

Integral on a polar rectangle:
$$\iint_D f(x,y)dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\underline{r}drd\theta$$
(Do not forget the extra r!)

Integral on a more general polar region:
$$\iint_D f(x,y)dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\underline{r}drd\theta$$
(Do not forget the extra r!)

Examples

Example 8:

1. $\iint_R (3x+4y)dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.
2. $\iint_R \tan^{-1}(y/x)dA$, where $R=\{(x,y): 1\le x^2+y^2\le 4, 0\le y\le x\}$.
3. Find the area of the region enclosed by the curve $r=4+3\cos\theta$.
4. Find the volume of the solid bounded by the paraboloids $z=3x^2+3y^2$ and $z=4-x^2-y^2$.
5. $\int_{-\infty}^{\infty} e^{-x^2}dx$.
6. Let R be the region inside the circle $x^2+y^2=4$ and above the line $y=\sqrt{3}$. Evaluate
$$\iint_R \frac{y}{x^2+y^2}dA.$$

Note: we can still not solve $\int_0^x e^{-s^2}ds$.

$f(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is the standard normal distribution or the unit normal distribution denoted by $\mathcal{N}(0,1)$

$F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-s^2/2}ds$ is the corresponding cumulative distribution function.

Now we have proved that $\lim_{x\to\infty} F(x)=1$.