Double Integrals on a rectangle

Double integral on R: $\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij},y_{ij})\Delta A$

Calculation of Double integral on R

$$\iint_R f(x,y)dA = \int_c^d \left[\int_a^b f(x,y)dx\right] dy = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

Question: What if the domain D of the function is not a rectangle?

Definition: similar to rectangle (a little cheating)
Use a bigger rectangle R to cover R, and define a function $g(x,y)$ on R such that $g(x,y) = f(x,y)$ if $(x,y) \in D$, and $g(x,y) = 0$ if $(x,y) \notin D$. Then define the integral of $f(x,y)$ on D to be the integral of $g(x,y)$ on R.

How to solve Double Integrals on a 2-D domain
Step 1 Find one (or two) algebraic presentation of the domain
a form of $\{(x,y) \in \mathbf{R}^2 : a \le x \le b, g_1(x) \le y \le g_2(x)\}$ or
a form of $\{(x,y) \in \mathbf{R}^2 : a \le y \le b, g_1(y) \le x \le g_2(y)\}$

Example 3
(1) Region D bounded by $y = 0$, $y = x^2$ and $x = 1$;
(2) Region D bounded by a triangle with vertices $(0,0)$, $(3,4)$ and $(6,0)$;
(3) Region D bounded by a circle with radius $R = 3$ and center $(2,5)$.

Solve Double Integrals on a 2-D domain

1. **Step 1** Find one (or two) algebraic presentation of the domain
   a form of $\{(x, y) \in \mathbf{R}^2 : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ or
   a form of $\{(x, y) \in \mathbf{R}^2 : a \leq y \leq b, g_1(y) \leq x \leq g_2(y)\}$
2. **Step 2**: Represent the integral on D in a form of iterated integrals
3. **Step 3**: Solve the iterated integrals

Example 4

1. $\iint_D xydA$, where the region D is bounded by $y = 0$, $y = x^2$ and $x = 1$;
2. $\iint_D x^2 dA$ where the region D is bounded by a triangle with vertices $(0, 0)$, $(3, 4)$ and $(6, 0)$;
3. $\iint_D ydA$ where the region D is in the first quadrant bounded by the parabolas $x = y^2$ and $x = 8 - y^2$.

Volume

Volume over a solid bounded by two surfaces $z = f(x, y)$ and $z = g(x, y)$ over a region D in xy-plane is

$$\iint_D [g(x, y) - f(x, y)]dA$$

(assume that $g(x, y) > f(x, y)$ for $(x, y) \in D$.)

Example 5

1. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.
2. Find the volume of the solid that is bounded by the planes $z = x$, $y = x$, $x + y = 2$ and $z = 0$.

Example 6 (Reverse the order of integrals)

1. Sketch the region of integration and change the order of integration:
$$\int_0^2 \int_{x^2}^4 f(x, y)dydx, \int_0^9 \int_0^{\sqrt{9-y}} f(x, y)dxdy$$
2. Evaluate the integral: $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1}dxdy$