Integrals

The definite integral is area under the curve when $f(x) \geq 0$

$$\int_a^b f(x)\,dx = \text{area bounded by } y = f(x), \; y = 0, \; x = a \text{ and } x = b$$

When $f(x)$ can be both positive and negative, the definite integral is the “signed” area

Definition: $$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(y_i)\Delta x \text{ (limit of Riemann sum)}$$

where $\Delta x = (b - a)/n, \; x_0 = a, \; x_1 = x_0 + \Delta x, \; \cdots, \; x_{k+1} = x_k + \Delta x, \; \cdots, \; x_n = b$, and $x_{i-1} \leq y_i \leq x_i$. 
Geometry of integral

Geometric idea: use rectangle with similar height to approximate irregular but almost rectangular shape

Solving the integrals: (Fundamental Theorem of Calculus)
\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a), \text{ and } F(x) \text{ is an antiderivative of } f(x). \]
\[ F(x) = \int f(x) \, dx \] is also called indefinite integral of \( f(x) \).
Elementary integral formulas

\[
\begin{align*}
\int x^n \, dx &= \frac{1}{n+1}x^{n+1} + C \\
\int \frac{1}{x} \, dx &= \ln |x| + C \\
\int e^x \, dx &= e^x + C \\
\int a^x \, dx &= \frac{1}{\ln a}a^x + C \\
\int \frac{1}{\sqrt{1-x^2}} \, dx &= \sin^{-1} x + C \\
\int \frac{1}{1+x^2} \, dx &= \tan^{-1} x + C
\end{align*}
\]

Other techniques:

(a) substitution \( \int f(u(x))u'(x) \, dx = \int f(u) \, du; \)

(b) integral by parts \( \int udv = uv - \int vdu; \)

(c) others (trigonometric substitution, partial fraction) probably not using (c) much in this class
Integrals on a rectangle

Domain: $R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\}$.
Function: $z = f(x, y) \geq 0$ defined on $R$
Geometric object: $S = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq f(x, y), (x, y) \in R\}$.

Question: Volume of $S$?

**Definition**: $\Delta x = (b - a)/n$, $\Delta y = (d - c)/m$, $\Delta A = \Delta x \cdot \Delta y$

$x_0 = a$, $x_1 = x_0 + \Delta x$, $\cdots$, $x_{k+1} = x_k + \Delta x$, $\cdots$, $x_n = b$;

$y_0 = a$, $y_1 = y_0 + \Delta y$, $\cdots$, $y_{k+1} = y_k + \Delta y$, $\cdots$, $y_m = d$;

$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ (a small rectangle), Pick a point $(x_{ij}, y_{ij})$ in $R_{ij}$

Double Riemann sum: $\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}, y_{ij}) \Delta A$

Double integral on $R$: $\iint_{R} f(x, y) dA = \lim_{m,n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}, y_{ij}) \Delta A$

**Approximate integrals**: Calculate the Riemann sum: which point to pick?
Midpoint rule: pick the center of the rectangle;
Corner rule: pick one of the corner points

Calculate double integral by volume: $\iint_{R} (5 - x) dA$, where $R = [0, 5] \times [0, 3]$
(not very useful)
Solving double integral on rectangle

Iterated integral: \[ \int_c^d \left[ \int_a^b f(x, y) \, dx \right] \, dy \]

First integrate with respect to \( x \) (treat \( y \) as a constant)
Then integrate with respect to \( y \)

**Fubini's Theorem:** If \( f(x, y) \) is continuous on the rectangle \( R = [a, b] \times [c, d] \), then

\[
\int\int_R f(x, y) \, dA = \int_c^d \left[ \int_a^b f(x, y) \, dx \right] \, dy = \int_a^b \left[ \int_c^d f(x, y) \, dy \right] \, dx
\]

**Examples:**

1. \[ \int\int_R \frac{xy^2}{x^2 + 1} \, dA, \ R = [0, 1] \times [-3, 3] \]
2. \[ \int\int_R \frac{x}{x^2 + y^2} \, dA, \ R = [1, 2] \times [0, 1] \]
3. Find the volume of the solid enclosed by the elliptic paraboloid \( x^2/4 + y^2/9 + z = 1 \) and above the rectangle \( R = [-1, 1] \times [-2, 2] \).
4. Find the average value of \( f(x, y) = e^y \sqrt{x} + e^y \) over \( R = [0, 4] \times [0, 1] \).