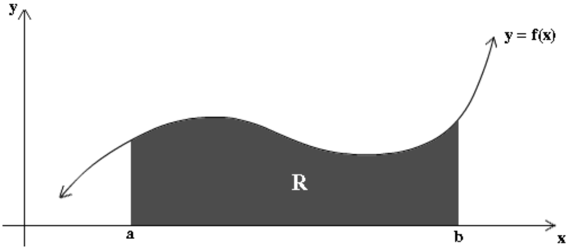
Integrals

The definite integral is area under the curve when $f(x) \geq 0$

$$\int_a^b f(x)dx = \text{area bounded by } y = f(x),\ y = 0,\ x = a \text{ and } x = b$$



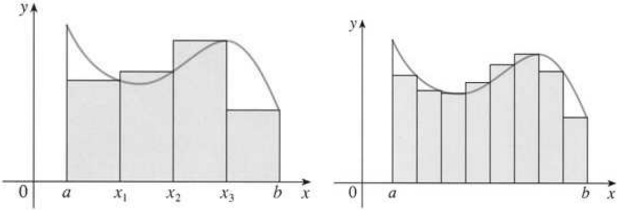
When $f(x)$ can be both positive and negative, the definite integral is the "signed" area

Definition: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(y_i)\Delta x$ (limit of Riemann sum)

where $\Delta x = (b-a)/n$, $x_0 = a$, $x_1 = x_0 + \Delta x$, $\cdots$, $x_{k+1} = x_k + \Delta x$, $\cdots$, $x_n = b$, and $x_{i-1} \leq y_i \leq x_i$.

Geometry of integral

Geometric idea: use rectangle with similar height to approximate irregular but almost rectangular shape



(b) $n = 4$ (c) $n = 8$

Solving the integrals: (Fundamental Theorem of Calculus)

$\int_a^b f(x)dx = F(b) - F(a)$, and $F(x)$ is an antiderivative of $f(x)$.

$F(x) = \int f(x)dx$ is also called indefinite integral of $f(x)$.

Elementary integral formulas

$\int x^n \, dx = \frac{1}{n+1} x^n + C$	$\int \sin x \, dx = -\cos x + C$
$\int \frac{1}{x} \, dx = \ln\|x\| + C$	$\int \cos x \, dx = \sin x + C$
$\int e^x \, dx = e^x + C$	$\int \sec^2 x \, dx = \tan x + C$
$\int a^x \, dx = \frac{1}{\ln a} a^x + C$	
$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

Other techniques:

(a) substitution $\int f(u(x))u'(x)dx = \int f(u)du$;

(b) integral by parts $\int udv = uv - \int vdu$;

(c) others (trigonometric substitution, partial fraction)

probably not using (c) much in this class

Integrals on a rectangle

Domain: $R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\}$.
Function: $z = f(x, y) \geq 0$ defined on R
Geometric object: $S = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq f(x, y), (x, y) \in R\}$.
Question: Volume of S?

Definition: $\Delta x = (b-a)/n$, $\Delta y = (d-c)/m$, $\Delta A = \Delta x \cdot \Delta y$
$x_0 = a$, $x_1 = x_0 + \Delta x$, $\cdots$, $x_{k+1} = x_k + \Delta x$, $\cdots$, $x_n = b$;
$y_0 = a$, $y_1 = y_0 + \Delta y$, $\cdots$, $y_{k+1} = y_k + \Delta y$, $\cdots$, $y_m = d$;
$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ (a small rectangle), Pick a point (x_{ij}, y_{ij}) in R_{ij}

Double Riemann sum: $\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}, y_{ij}) \Delta A$

Double integral on R: $\iint_R f(x, y) dA = \lim_{m,n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}, y_{ij}) \Delta A$

Approximate integrals:
Calculate the Riemann sum: which point to pick?
Midpoint rule: pick the center of the rectangle;
Corner rule: pick one of the corner points

Calculate double integral by volume: $\iint_R (5 - x) dA$, where $R = [0, 5] \times [0, 3]$

(not very useful)

Solving double integral on rectangle

Iterated integral: $\int_c^d \left[\int_a^b f(x,y)dx\right] dy$

First integrate with respect to x (treat y as a constant)

Then integrate with respect to y

Fubini's Theorem: If $f(x,y)$ is continuous on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)dA = \int_c^d \left[\int_a^b f(x,y)dx\right] dy = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

Examples:

1. $\iint_R \frac{xy^2}{x^2+1} dA,\ R = [0,1] \times [-3,3]$
2. $\iint_R \frac{x}{x^2+y^2} dA,\ R = [1,2] \times [0,1]$
3. Find the volume of the solid enclosed by the elliptic paraboloid $x^2/4 + y^2/9 + z = 1$ and above the rectangle $R = [-1,1] \times [-2,2]$.
4. Find the average value of $f(x,y) = e^y\sqrt{x+e^y}$ over $R = [0,4] \times [0,1]$.