Constrained Optimization

Question: Find the extreme values of $f(x, y)$ when $g(x, y) = k$

Let $g(x, y) = k$ have a parameterized equation $(x(t), y(t))$.
Then $g(x(t), y(t)) = k$, and $g_x x' + g_y y' = 0$ from differentiation.
The extreme values of $F(t) = f(x(t), y(t))$ are achieved at critical points where $F'(t) = 0$. $F'(t) = f_x x' + f_y y' = 0$.
Therefore $\langle g_x, g_y \rangle \cdot \langle x', y' \rangle = 0$ and $\langle f_x, f_y \rangle \cdot \langle x', y' \rangle = 0$.
So $\langle g_x, g_y \rangle$ is parallel to $\langle f_x, f_y \rangle$, or $\nabla f = \lambda \nabla g$.

Solve Constrained Optimization Problem

1. Solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = k$.
   (λ is called the Lagrange Multiplier of the problem)
2. Compare all solutions of the above equation, the smallest one is the absolute minimum value, and the largest one is the absolute maximum value.

Examples

Examples: Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

1. $f(x, y) = 3x + y$; $x^2 + y^2 = 10$.
2. $f(x, y) = x^2 + y^2$; $3x + y = 10$.
3. Find the Absolute maximum and minimum values of $f(x, y) = 2x^2 + 3y^2 - 4x + 6y - 5$ on $D = \{(x, y) : x^2 + y^2 \leq 16\}$.
4. Find the point on the plane $x - y + z = 4$ that is closest to the point $(1, 2, 3)$.
5. Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest to and farthest from the point $(1, 2, 3)$.