# Optimization problem: minimum and maximum

1-D: $y = f(x)$,    critical point: $f'(x) = 0$
**Theorem A**: (1st derivative test) If $x = x_0$ is a local maximum or local minimum of $y = f(x)$, then $f'(x_0) = 0$ (so $x_0$ is a critical point).
**Theorem B**: (2nd derivative test) Suppose that $f'(x_0) = 0$. If $f''(x_0) > 0$, then $x_0$ is a local minimum; and if $f''(x_0) < 0$ , then $x_0$ is a local maximum. When $f''(x_0) = 0$, we need additional information.
Absolute minimum or maximum on $[a, b]$: either a local one, or at boundary points $x = a, b$.
**Theorem C**: A continuous function $y = f(x)$ always achieves absolute minimum and maximum on a closed interval $[a, b]$.

2-D: $z = f(x, y)$,    critical point: $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = 0$
**Theorem 1**: (1st derivative test) If $(x_0, y_0)$ is a local maximum or local minimum of $z = f(x, y)$, then $\nabla f(x_0, y_0) = 0$, or $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.
Local minimum point: $f(x_0, y_0) \leq f(x, y)$ for all nearby $(x, y)$
Local maximum point: $f(x_0, y_0) \geq f(x, y)$ for all nearby $(x, y)$
Saddle point: a critical point which is not a local maximum nor local minimum, and it is local maximum in one direction, local minimum in another one.

# Second-derivative test in 2-D

**Theorem 2**: (2nd derivative test)
Let $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2$
(a) If $f_{xx}(x_0, y_0) > 0$, $f_{yy}(x_0, y_0) > 0$, and $D > 0$, then $(x_0, y_0)$ is a local minimum;
(b) If $f_{xx}(x_0, y_0) < 0$, $f_{yy}(x_0, y_0) < 0$, and $D > 0$, then $(x_0, y_0)$ is a local maximum;
(c) If $D < 0$, then $(x_0, y_0)$ is a saddle point;
(d) If $D = 0$ or $f_{xx} = 0$, $f_{yy} = 0$, other type or undetermined.

**Example**: $f(x, y) = x^2 + y^2$ (local minimum at $(0, 0)$);
$f(x, y) = -x^2 - y^2$ (local maximum at $(0, 0)$);
$f(x, y) = x^2 - y^2$ (saddle point at $(0, 0)$)

function: $f(x, y)$, Gradient (vector): $\nabla f(x, y) = (\partial f/\partial x, \partial f/\partial y)$
Hessian (matrix): $H(f) = \begin{pmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{pmatrix}$,
$T = trace(H) = f_{xx} + f_{yy}$, $D = det(H) = f_{xx} f_{yy} - [f_{xy}]^2$

**Second-derivative test in 2-D**:
(a) If $T > 0$, and $D > 0$, then $(x_0, y_0)$ is a local minimum;
(b) If $T < 0$, and $D > 0$, then $(x_0, y_0)$ is a local maximum;
(c) If $D < 0$, then $(x_0, y_0)$ is a saddle point.
**Example 1:** Find and classify critical points
(1) $f(x, y) = x^3 y + 12x^2 - 8y$, (2) $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$
(3) Find the shortest distance from the point $(0, 1, 1)$ to the plane $x - 2y + 3z = 6$.

# Absolute maximum and minimum

**Theorem 3**: A continuous function $f(x, y)$ defined on a closed bounded subset $D$ of $\mathbb{R}^2$ has an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$.

**How to find?**

1. Find the values of $f$ at all critical points in $D$;
2. Find the maximum and minimum on the boundary of $D$;
3. Compare all these values, the smallest one is the absolute minimum value, and the largest one is the absolute maximum value.

**Example 2:**

1. Find the Absolute maximum and minimum values of $f(x, y) = 4x + 6y - x^2 - y^2$ on $D = \{(x, y) : 0 \leq x \leq 4, 0 \leq y \leq 5\}$.
2. Find the point on the line $L_1$: $x = 1 + t$, $y = 1 + 6t$, $z = 2t$ that is closest to the point $(1, 2, 3)$.
3. Find the distance between the skew lines $L_1$: $x = 1 + t$, $y = 1 + 6t$, $z = 2t$, and $L_2$: $x = 1 + 2s$, $y = 5 + 15s$, $z = -2 + 6s$.
4. The base of an aquarium with the given volume $V$ is made of slate and the sides are made of glass. If the slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.
5. Find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible.
6. Find the Absolute maximum and minimum values of $f(x, y) = 2x^2 + 3y^2 - 4x + 6y - 5$ on $D = \{(x, y) : x^2 + y^2 \leq 16\}$. (How to solve: $\max\{f(x, y) : x^2 + y^2 = 16\}$?)