Directional derivative

Directional derivative: $z = f(x, y)$

$f_x(x, y)$: the derivative at x-direction; $f_y(x, y)$: the derivative at y-direction
derivative in other direction?

Consider a curve $(x(t), y(t))$ pass through $(x(0), y(0))$, then
$\frac{\partial f}{\partial t}(x(t), y(t)) = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$
This rate of change may depend on the velocity $\mathbf{v}(t) = \langle x'(t), y'(t) \rangle$, so if we take the unit velocity:
$\langle \frac{x'(0)}{\sqrt{[x'(0)]^2 + [y'(0)]^2}}, \frac{y'(0)}{\sqrt{[x'(0)]^2 + [y'(0)]^2}} \rangle$
instead of $\langle x', y' \rangle$

Directional derivative: for a function $z = f(x, y)$, the directional derivative in the direction of $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = f_x(x_0, y_0)a + f_y(x_0, y_0)b =$$
$\langle f_x, f_y \rangle \cdot \langle a, b \rangle$

f_x is the directional derivative in $\mathbf{u} = \langle 1, 0 \rangle$, f_y is the directional derivative in $\mathbf{u} = \langle 0, 1 \rangle$

$\langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$ is the **gradient vector** at (x_0, y_0)
Notation: $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$

Examples

1. Find the directional derivative of $f(x, y) = 5xy^2 - 4x^3y$ at $(x, y) = (1, 2)$ in the direction of $\mathbf{u} = \langle 5/13, 12/13 \rangle$
2. Find the directional derivative of $f(x, y) = x\sin(xy)$ at $(x, y) = (2, 0)$ in the direction which has an angle $\theta = \pi/3$ with positive x-axis
3. Find the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ at $P(1, 2, 3)$ in the direction of the origin.

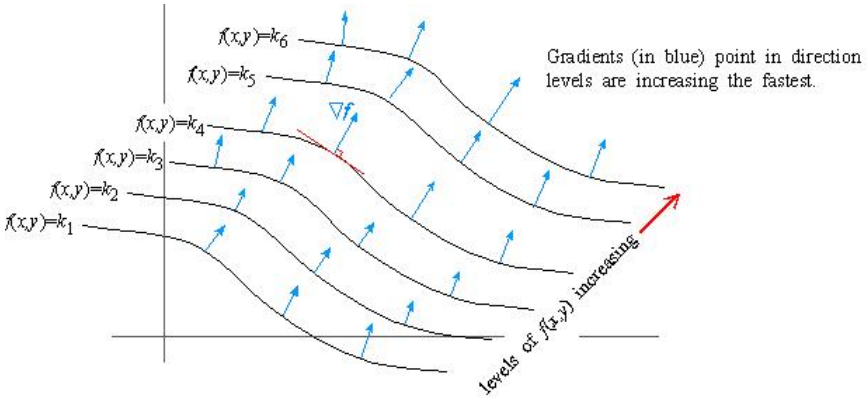
Geometry of directional derivative

$D_{\mathbf{u}}f(x_0, y_0) = \langle f_x, f_y \rangle \cdot \langle a, b \rangle = |\nabla f| \cdot \cos\theta$
(θ is the angle between ∇f and $\mathbf{u}$)

- when $\mathbf{u}$ is in the same direction as ∇f, $D_{\mathbf{u}}f(x_0, y_0) = |\nabla f|$, the maximum rate of change in any direction. Thus the direction of ∇f is where $f(x, y)$ increases the fastest.
- when $\mathbf{u}$ is in the opposite direction as ∇f, $D_{\mathbf{u}}f(x_0, y_0) = -|\nabla f|$, the minimum rate of change in any direction. Thus the direction of $-\nabla f$ is where $f(x, y)$ decreases the fastest.
- when $\mathbf{u}$ is perpendicular to ∇f, then $D_{\mathbf{u}}f(x_0, y_0) = 0$. These two directions ($\theta = \pm\pi$) are where f neither increases nor decreases. In fact these are the directions of tangent lines of level curve $f(x, y) = c = f(x_0, y_0)$.

<u>Gradient is perpendicular to level curves</u>
Tangent line of level curve:
$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$
Tangent plane of level surface:
$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$
Normal line of the level surface: $\dfrac{x - x_0}{f_x(x_0, y_0, z_0)} = \dfrac{y - y_0}{f_y(x_0, y_0, z_0)} = \dfrac{z - z_0}{f_z(x_0, y_0, z_0)}$

Geometry of directional derivative

Gradient is natural: from level curve $f(x, y) = c_1$ to $f(x, y) = c_2$ ($c_1 > c_2$), if the shortest path is $(x(t), y(t))$, then $\langle x'(t), y'(t)\rangle$ is always the direction of $-\nabla f$ (the steepest descent direction)
fastest descent direction: the path of the creek in mountain falling from the peak to the bottom
Examples:

1. In what direction are the levels of $f(x, y) = x^3 y$ increasing the fastest at the point $(2, 3)$? What is the maximum rate of change?
2. Find the tangent plane and normal line of the surface $x^2 - 2y^2 + z^2 + yz = 2$