Plane and Space

Plane and Space: a plane is two-dimensional, and each point in the plane can be described by a pair of number (x, y); a space is three dimensional, and each point in the space can be described by a triple (x, y, z).

Plane: $(0, 0)$ is the origin, x-axis is where $y = 0$, y-axis is where $x = 0$, and there are four quadrants divided by x-axis and y-axis. (the first quadrant is where $x > 0$ and $y > 0$.)

Space: $(0, 0, 0)$ is the origin; x-axis is where $y = 0$ and $z = 0$, y-axis is where $x = 0$ and $z = 0$, z-axis is where $x = 0$ and $y = 0$; xy-plane is where $z = 0$, xz-plane is where $y = 0$, yz-plane is where $x = 0$; and there are eight octants divided by xy-plane, xz-plane and yz-plane. (the first octant is where $x > 0$, $y > 0$ and $z > 0$.)

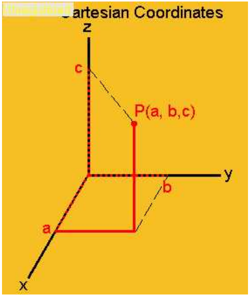
Line: $\mathbb{R} = \{x \in (-\infty, \infty)\}$ ($\in$ means belong to)
Plane: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$
Space: $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$

A **point** in $\mathbb{R}^3$ has the coordinate (x, y, z) to denote its location.

3-d Cartesian coordinate system

Right-hand rule: (textbook) if you curl your right hand around from positive x-axis to positive y-axis, then your thumb points in the positive z-axis.

Right-hand rule: (Prof Shi) if you stretch the thumb, index finger and middle finger of your right hand, thumb points in the positive x-axis, index finger points in the positive y-axis, then your middle finger points in the positive z-axis.



Point sets in the space

$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$: the whole space
an equation $f(x, y, z) = 0$ in $\mathbb{R}^3$: a surface in $\mathbb{R}^3$
(an equation $f(x, y) = 0$ in $\mathbb{R}^2$: a curve in $\mathbb{R}^2$)

Example 1: Describe the surfaces represented by the equations.
(a) $x = 0$; (b) $y = -2$; (c) $x - z = 0$; (d) $y = x^2$; (e) $x^2 + y^2 = 4$.

Example 2: Describe the following sets in $\mathbb{R}^3$:
(a) $y \geq 2$; (b) $1 \leq x \leq 4$; (c) $x = 1$ and $y = 4$; (d) $x^2 + y^2 < 4$;
(e) $x^2 + y^2 = 4$ and $z = 2$; (f) $x^2 + y^2 + z^2 = 4$;
(g) $x^2 + y^2 + z^2 < 4$; (h) $x^2 + y^2 + z^2 > 4$.

We will learn a general equation of a plane and a line in Section 12.5

Distance formula and sphere

Point: $P(x, y, z)$ means a point with coordinate (x, y, z) which we call P

Distance formula: distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 3: $P = (-2, 3, -5)$
(a) Find the distance of P to $Q(1, 8, -4)$;
(b) Find the distance of P to xy-plane;
(c) Find the distance of P to x-axis.

Sphere: a sphere with center at $P(a, b, c)$ and radius r is the set of all points whose distance to P is r:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Example 4:
(a) Find the intersection of $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 4$ with yz-plane;
(b) Describe the set of points $\{(x, y, z) : 1 \leq x^2 + y^2 + z^2 \leq 9\}$;
(c) Find the center and radius of the sphere $x^2 + y^2 + z^2 = 4x - 2y + 10$.

Vector

Vector is a quantity that has both magnitude and direction.
Examples: velocity, force......
Vector can be represented by two, or three, or more numbers

Scalar is a quantity that has magnitude and sign.
Example: distance, work......
Scalar can be represented by one number (positive or negative)

A vector in 2-d or 3-d space has an initial point A, and a terminal point B; but it does not depend on the initial or terminal point.

Example 5: Let $A(1,2,3)$ and $B(-3,4,6)$ be two points. Then vector $\overrightarrow{AB}$ starts from $A(1,2,3)$ and ends at $B(-3,4,6)$, and $\overrightarrow{AB} = \langle(-3)-1, 4-2, 6-3\rangle = \langle -4, 2, 3\rangle$ (like $B - A$). But another vector from $C = (0,0,0)$ to $D = (-4,2,3)$ is $\overrightarrow{CD} = \langle -4, 2, 3\rangle$, so $\overrightarrow{AB} = \overrightarrow{CD}$ although their initial and terminal points are different.

Zero vector $\mathbf{0}$ or $\overrightarrow{0}$: initial point same as terminal point, length 0. It is the only vector with no specific direction.

Negative vector of $\overrightarrow{AB}$ is $\overrightarrow{BA}$: switch the initial point and terminal point.
$\overrightarrow{AB} = -\overrightarrow{BA}$.

Definition of vector

Given $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the vector $\overrightarrow{P_1P_2}$ is

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

(Note $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_1}$ are different—they are on the opposite directions, but have same length.)

Length (or magnitude) of $\overrightarrow{P_1P_2}$ is

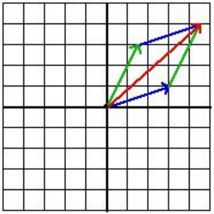
$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

A vector $\langle a, b, c \rangle$ has x-component a, y-component b and z-component c.

For a point $P(a, b, c)$, its **position vector** is $\langle a, b, c \rangle$. For a vector $\langle a, b, c \rangle$, there are many different representations like $\overrightarrow{P_1P_2}$ for some $P_1, P_2 \in \mathbb{R}^3$.

Addition of vectors

Addition of vectors:
$\langle a, b, c\rangle + \langle d, e, f\rangle = \langle a+d, b+e, c+f\rangle$
$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ (Triangle law) (parallelogram law)

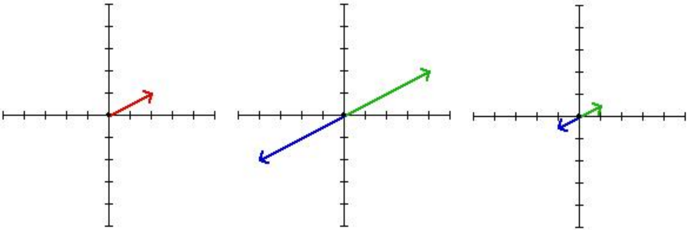


Blue $\overrightarrow{AB}$, Green $\overrightarrow{BC}$, red $\overrightarrow{AC}$

Difference of vectors:
$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$

Scalar multiplication of a vector

multiply a number to a vector
$k\langle a, b, c\rangle = \langle ka, kb, kc\rangle$
$k\vec{u}$ has the same direction as $\vec{u}$ if $k > 0$,
opposite direction if $k < 0$



(a) red $\vec{u}$; (b) green $2\vec{u}$, blue $-2\vec{u}$; (c) green $0.5\vec{u}$, blue $-0.5\vec{u}$.

Example 6: Given vectors $\vec{u}$ and $\vec{v}$. Draw $2\vec{u} + 3\vec{v}$, and $\vec{u} - 2\vec{v}$.

Properties of vectors

(addition and scalar multiplication)
$\vec{u}+\vec{v}=\vec{v}+\vec{u}$, $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$, $\vec{u}+\vec{0}=\vec{0}+\vec{u}$, $\vec{u}+(\vec{-u})=\vec{0}$,
$c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$, $(a+b)\vec{u}=a\vec{u}+b\vec{u}$, $(cd)\vec{u}=c(d\vec{u})$, $1\vec{u}=\vec{u}$

Unit vector : length $=1$
$\vec{v}=\frac{1}{|\vec{u}|}\vec{u}$ is a unit vector which has the same direction as $\vec{u}$. The **direction** of any vector $\vec{u}$ can be understood as the unit vector $\vec{v}=\frac{1}{|\vec{u}|}\vec{u}$ which has the same direction as $\vec{u}$.

Standard representation of a vector:
Each vector can be written as $\mathbf{v}=|\mathbf{v}|\cdot\frac{\mathbf{v}}{|\mathbf{v}|}$, with $|\mathbf{v}|$ the length and $\frac{\mathbf{v}}{|\mathbf{v}|}$ the (unit) direction vector.

Three special vectors: (standard basis vectors)
$\mathbf{i}=\langle 1,0,0\rangle$, $\mathbf{j}=\langle 0,1,0\rangle$, and $\mathbf{k}=\langle 0,0,1\rangle$
($\mathbf{i}$: positive x direction unit vector, $\mathbf{j}$: positive y direction unit vector, $\mathbf{k}$: positive z direction unit vector)

Second representation of a vector:
$\langle a,b,c\rangle=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$

Examples

Example 7: $\vec{u} = 3\mathbf{i} - 2\mathbf{k}$, $\vec{v} = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Find $3\vec{u} - 4\vec{v}$, and the unit vector which has the opposite direction as $3\vec{u} - 4\vec{v}$.

Example 8: (problem 12.2.7) Suppose that $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$. Let the midpoint of QR be S, let $\mathbf{c} = \overrightarrow{PS}$, and let $\mathbf{d} = \overrightarrow{SR}$. Express $\mathbf{c}$ and $\mathbf{d}$ in the terms of $\mathbf{a}$ and $\mathbf{b}$.

Example 9: Ropes 3m and 5m in length are fasten to a holiday decoration that is suspended over a town square. The decoration has a mass of 5kg. The ropes, fasten at different heights, make angles of 52° and 40° with the horizontal. Find the tension in each wire and the magnitude of each tension.

Example 10: Let $\vec{b} = \langle 1, 1 \rangle$ and $\vec{c} = \langle -1, 2 \rangle$. Pick any vector $\vec{a}$. Show that $\vec{a} = s\vec{b} + t\vec{c}$ for some $s, t \in \mathbb{R}$.