Lagrange Multiplier

Math 212

Here is the rigorous mathematical result for the Lagrange Multiplier Method. The proof can be found in some other books on Analysis.

Theorem 1 (Lagrange Multiplier Method)**.** *Let $S = \{(x,y,z) : g(x,y,z) = c\}$ be a differentiable surface in $\mathbf{R}^3$ where $c \in \mathbf{R}$, and let $w = f(x,y,z)$ be a continuously differentiable function defined in $\mathbf{R}^3$. If the restriction of w on S achieves a local maximum (or local minimum) value at $(x_0, y_0, z_0) \in S$, then there exists $\lambda \in \mathbf{R}$ such that $\nabla f(x_0, y_0, z_0) = \lambda g(x_0, y_0, z_0)$ and $g(x_0, y_0, z_0) = c$. Moreover if S is bounded closed subset in $\mathbf{R}^3$, then the restriction of w on S must achieve an absolute minimum value and an absolute maximum value on S, and the absolute minimum/maximum is achieved at either a point such that $\nabla f(x_0, y_0, z_0) = \lambda g(x_0, y_0, z_0)$ and $g(x_0, y_0, z_0) = c$, or is achieved on a boundary point of S.*

Example 1. Find the point (x,y) on the curve $17x^2 + 12xy + 8y^2 = 100$ which is closest and farthest to the origin $(0,0)$.

Solution. The square of the distance from (x,y) to $(0,0)$ is $f(x,y) = x^2 + y^2$, and it is under the constraint $g(x,y) = 17x^2 + 12xy + 8y^2 = 100$. Then the Lagrange Multiplier Method gives

$$2x = \lambda(34x + 12y), \tag{1}$$

$$2y = \lambda(12x + 16y), \tag{2}$$

$$17x^2 + 12xy + 8y^2 = 100. \tag{3}$$

Dividing (1) by (2), we get $\dfrac{x}{y} = \dfrac{34x + 24y}{24x + 16y} = \dfrac{17x/y + 6}{6x/y + 8}$. Let $z = x/y$, then z satisfies $z(6z+8) = 17z + 6$, $6z^2 - 9z - 6 = 0$, $2z^2 - 3z - 2 = 0$. Thus $z = -1/2$ or $z = 2$.

Case 1: $z = -1/2$, so $x = -1/2y$. From (3), we get $(17/4 - 6 + 8)y^2 = 100$, $(25/4)y^2 = 100$, $y^2 = 16$, $y = \pm 4$ and $x = \mp 2$. This gives two critical points: $(x,y) = (-2,4)$ and $(2,-4)$.

Case 2: $z = 2$, so $x = 2y$. From (3), we get $(68 + 24 + 8)y^2 = 100$, $100y^2 = 100$, $y^2 = 1$, $y = \pm 1$ and $x = \pm 2$. This gives two critical points: $(x,y) = (2,1)$ and $(-2,-1)$.

Now $f(\pm 2, \pm 1) = 5$ and $f(\mp 2, \pm 4) = 20$. Hence $(x,y) = (2,1)$ and $(-2,-1)$ are the points which are closest to the origin, and $(x,y) = (-2,4)$ and $(2,-4)$ are the points which are farthest to the origin.

(Alternative way suggested by I. C.) Note that (1) and (2) can be viewed as an eigenvalue problem: $\begin{pmatrix} 17 & 6 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$. Let $\lambda^{-1} = \theta$. Then $(17-\theta)(8-\theta) - 36 = 0$, $\theta^2 - 25\theta + 100 = 0$. So $\theta = 20$ or $\theta = 5$, and $\lambda = 1/5$ or $1/20$. Solving eigenvector and using the constraint, one can also get $(x,y) = (2,1)$ and $(-2,-1)$, $(x,y) = (-2,4)$ and $(2,-4)$. □

Example 2. *Find the point on the curve $z^2 + 3x - xy = 9$ which is nearest to $(0,0,0)$.*

Solution 1. The square of the distance from (x,y,z) to $(0,0,0)$ is $f(x,y,z) = x^2 + y^2 + z^2$, and it is under the constraint $g(x,y,z) = z^2 + 3x - xy = 9$. We substitute $z^2 = 9 - 3x + xy$ from the constraint to f, then $f(x,y,z) = x^2 + y^2 + 9 - 3x + xy$. Then we consider the new function $F(x,y) = x^2 + y^2 + 9 - 3x + xy$ under a new constraint $9 - 3x + xy \geq 0$ as $z^2 \geq 0$. So we look for the absolute minimum of $F(x,y)$ in the region $D = \{(x,y) : 9 - 3x + xy \geq 0\}$. From the method in Section 14.7, it is achieved either at a critical point, or a boundary point. For critical point, we find $F_x = 2x - 3 + y = 0$ and $F_y = 2y + x = 0$, and we solve $(x, y) = (2, -1)$ to be the only critical point. One can verify $(2, -1) \in D$ so it is valid. For the value of $F(x, y)$ on the boundary $B = \{(x, y) : 9 - 3x + xy = 0\}$, we use another substitution $y = (3x - 9)/x = 3 - 9/x$. Then $G(x) = F(x, 3 - 9/x) = x^2 + (3 - 9/x)^2 = x^2 - 54/x + 81/x^2 + 9$. From $G'(x) = 2x + 54/x^2 - 162/x^3 = 0$, we get $x^4 + 54x - 81 = 0$. By using a computer solver, one gets $x_1 = -3.66223$ ($y_1 = 5.45752$), and $x_2 = 2.17348$ ($y_2 = -1.14083$).

Now we evaluate $f(x, y, z)$ at $P_1 = (-3.66223, 5.45752, 0)$, $P_2 = (2.17348, -1.14083)$, and $P_3 = (2, -1, \pm 1)$, we get $f(P_1) > 30$, $f(P_2) = 6.025$, and $f(P_3) = 6$. Note that when $(x, y, z) = (1, n, \pm\sqrt{6 + n})$, then $g(x, y, z) = 9$, and $f(x, y, z) = 7 + n + n^2 \to \infty$ when $n \to \infty$, so f has no maximum value on $z^2 + 3x - xy = 9$, but f has a minimum value as $f \geq 0$. Then $f(P_3) = 6$ is the absolute minimum. □

Solution 2. One can also use Lagrange Multiplier Method.

$$2x = \lambda(3 - y), \tag{4}$$
$$2y = \lambda(-x), \tag{5}$$
$$2z = \lambda(2z), \tag{6}$$
$$z^2 + 3x - xy = 9. \tag{7}$$

From (6), we get $z = 0$ or $\lambda = 1$.

Case 1: $z = 0$. Then (7) implies that $3x - xy = 9$, or $y = (3x - 9)/x = 3 - 9/x$. Dividing (4) by (5), we get $\dfrac{x}{y} = \dfrac{3 - y}{-x}$, or $x^2 = -y(3 - y)$. Then $x^2 = -(3 - 9/x)(9/x)$, which reduces to $x^4 + 54x - 81 = 0$. So like Solution 1, we can get $P_1 = (-3.66223, 5.45752, 0)$, $P_2 = (2.17348, -1.14083)$.

Case 2: $\lambda = 1$. This gives $2x = 3 - y$ and $2y = -x$, which solves to $(x, y) = (2, -1)$, and $P_3 = (2, -1, \pm 1)$.

We use the same evaluation as in Solution 1 to get $f(P_3) = 6$ is the absolute minimum. □

Exercise

1. Find the shortest distance from the origin $(0, 0)$ to the hyperbola $x^2 + 8xy + 7y^2 = 225$.

2. Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.

3. A package in the shape of a rectangular box can be mailed by the US Postal Service if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108 in. Find the dimensions of the package with largest volume that can be mailed.