Bifurcation Diagrams of Coupled Schrödinger Systems

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Introduction

The nonlinear Schrödinger (NLS) equation is as follows:

$$i\psi_t + \Delta\psi + \gamma|\psi|^2\psi = 0 \qquad (1.1)$$

This is a canonical and universal equation that plays an important role in plasma physics, nonlinear optics, and condensed matter (i.e. Bose-Einstein condensate). We consider the two component system of the following form:

$$\begin{aligned} i\hbar\frac{\partial\phi_1}{\partial t} &= -\left(\frac{\hbar^2}{2m_1}\Delta + V_1(x) + \lambda_1|\phi_1|^2\right)\phi_1 + \beta|\phi_2|^2\phi_1 \\ i\hbar\frac{\partial\phi_2}{\partial t} &= -\left(\frac{\hbar^2}{2m_2}\Delta + V_2(x) + \lambda_2|\phi_2|^2\right)\phi_2 + \beta|\phi_1|^2\phi_2 \end{aligned} \qquad (1.2)$$

where $x \in R^n$ for n=1, 2, or 3; ϕ_j (j=1, 2) is the wave function of two interacting condensates; V_j is the trap potential; and interactions strengths λ_j and β are determined by scattering lengths. We look for a pulse-like soliton solution to (1.2) in the form:

$$\phi_j(t,x) = u_j(x)\exp(i\mu_j t/\hbar) \qquad (1.3)$$

and reduce equation (1.2) to a system of PDEs:

$$\begin{aligned} \frac{\hbar^2}{2m_1}\Delta u_1 + V_1(x)u_1 + \lambda_1|u_1|^2u_1 + \beta|u_2|^2u_1 &= \mu_1 u_1 \\ \frac{\hbar^2}{2m_2}\Delta u_2 + V_2(x)u_2 + \lambda_2|u_2|^2u_2 + \beta|u_1|^2u_2 &= \mu_2 u_2 \end{aligned} \qquad (1.4)$$

where we treat μ_j as a chemical potential. When V_j=0, the solutions to (1.4) are the canonical ground states. We consider such ground states of the form:

$$\begin{aligned} \Delta u_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 &= 0 \\ \Delta u_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 &= 0 \end{aligned} \qquad (1.5)$$

For $x \in R^n$ where λ_j, μ_j, $\beta > 0$, and n = 1, 2, or 3. We look for positive solutions of (1.5) when $|x| \to \infty$. Since such a solution is known to be radially symmetric and decay exponentially, we can consider:

$$\begin{aligned} &\Delta u_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0 \\ &\Delta u_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 \\ &u_1(x) > 0,\ u_2 > 0, \\ &u_1 \to 0,\ u_2(x) \to 0, \qquad |x| \to \infty. \end{aligned} \qquad (1.6)$$

Since solutions to (1.6) are radially symmetric, they satisfy the following:

$$\begin{aligned} &u_1'' + \frac{n-1}{r}u_1' - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0, \\ &u_2'' + \frac{n-1}{r}u_2' - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0, \\ &u_1'(0) = 0,\ u_1'(r) < 0,\ \lim_{r\to\infty} u_1(r) = 0, \\ &u_2'(0) = 0,\ u_2'(r) < 0,\ \lim_{r\to\infty} u_2(r) = 0. \end{aligned} \qquad (1.7)$$

In particular, this solution solves the system:

$$\begin{aligned} &u_1'' + \frac{n-1}{r}u_1' - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0, \\ &u_2'' + \frac{n-1}{r}u_2' - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0, \\ &u_1 = A > 0,\ u_1'(0) = 0, \\ &u_2 = B > 0,\ u_2'(0) = 0. \end{aligned} \qquad (1.8)$$

for $r > 0$. We consider (1.8) and its generalization numerically. Our results indicate for all parameters in (1.7) that the solution is unique. This is not proved for general coupled Schrodinger equations.

Mathematical Setting

We consider the initial value (1.8). Local existence and uniqueness can be proved via standard contraction mapping principle. We denote a solution to (1.8) by ($u_1(r;A,B)$), $u_2(r;A,B)$) or simply ($u_1(r)$, $u_2(r)$) when there is no confusion. The solution ($u_1(r)$, $u_2(r)$) can be extended to the maximal interval (0, R). Note that this includes the case that ($u_1(r)$, $u_2(r)$) is extended to $r = R$ and $u_1(R) = u_2(R) = 0$.

We look for two types of solutions. If

$$u_1(r) > 0,\ u_2(r) > 0,\ u_1'(r) < 0,\ u_2'(r) < 0,\ 0 < r < \infty, \qquad (2.1)$$

then ($u_1(r)$, $u_2(r)$) is a *ground state solution;* if

$$\begin{aligned} &u_1(r) > 0,\ u_2(r) > 0,\ u_1(r)\,; < 0,\ u_2' < 0,\ 0 < r < R, \\ &u_1(R) = u_2(R) = 0, \end{aligned} \qquad (2.2)$$

then ($u_1(r)$, $u_2(r)$) is a *crossing solution*. From previous work in the field, we know that any solution to (1.6) is radially symmetric and thus a solution to (1.8) satisfying (2.1). Define:

$$\begin{aligned} f(u_1,u_2) &\equiv -\lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2, \\ g(u_1,u_2) &\equiv -\lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2, \\ F(u_1,u_2) &= \tfrac{1}{2}(\lambda_1 u_1^2 + \lambda_2 u_2^2) + \tfrac{1}{4}(\mu_1 u_1^4 + 8\beta u_1^2 u_2^2 + \mu_2 u_2^4). \end{aligned} \qquad (2.3)$$

This is a gradient system. The set $\{f(u_1, u_2) = 0\}$ consists of the line $\{u_1 = 0\}$ and the ellipse $E_1 = \{\mu_1 u_1^2 + \beta u_2^2 = \lambda_1\}$, and the set $\{g(u_1, u_2) = 0\}$ and the ellipse $E_2 = \{\beta u_1^2 + \mu_2 u_2^2 = \lambda_2\}$. Let

$$\begin{aligned} \beta_1 &= \min\left\{\frac{\lambda_2}{\lambda_1}\mu_1, \frac{\lambda_2}{\lambda_1}\right\}, \\ \beta_2 &= \max\left\{\frac{\lambda_2}{\lambda_1}\mu_1, \frac{\lambda_2}{\lambda_1}\right\} \end{aligned} \qquad (2.4)$$

Then when $0 < \beta < \beta_1$, and $\beta > \beta_2$, E_1 and E_2 intersect exactly once in the first quadrant, and when $\beta 1 < \beta < \beta 2$, E_1 and E_2 do not intersect, hence one ellipse is inside the other. In the first case, $f = g = 0$ is a global minimum. According to the signs of f and g, we define the following region in $\mathbf{R}_+^{\,2}$:

$$\begin{aligned} I &= \{(u_1,u_2) \in \mathbf{R}_+^2 : f(u_1,u_2) > 0, g(u_1,u_2) > 0\}, \\ II &= \{(u_1,u_2) \in \mathbf{R}_+^2 : f(u_1,u_2) < 0, g(u_1,u_2) < 0\}, \\ III &= \{(u_1,u_2) \in \mathbf{R}_+^2 : f(u_1,u_2) < 0, g(u_1,u_2) > 0\}, \\ IV &= \{(u_1,u_2) \in \mathbf{R}_+^2 : f(u_1,u_2) > 0, g(u_1,u_2) < 0\}. \end{aligned} \qquad (2.5)$$

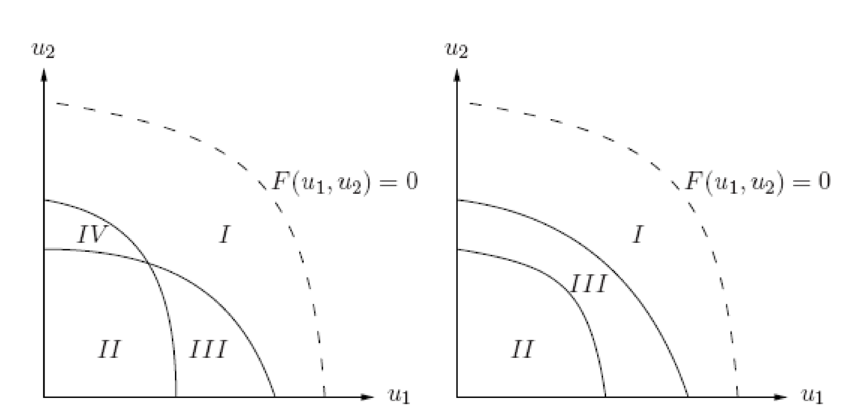

Figure 1: The regions of possible initial values (A, B): solid lines are $f(u_1, u_2) = 0$ and $g(u_1, u_2) = 0$ respectively; and the dashed line is $F(u_1, u_2) = 0$. (left): $0 < \beta < \beta_1$ and $\beta > \beta_2$; (right) $\beta_1 < \beta < \beta_2$.

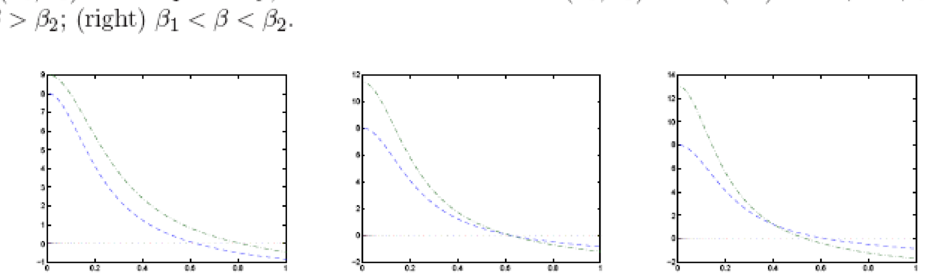

Figure 2: Solution curves of (1.10) when $n = 3$, $\mu_1 = \lambda_1 = \lambda_2 = 1$, $\mu_2 = 2$, $\beta = 0.01$. Initial values: $u(0) = 8$ in all three; (left) $v(0) = 9$ ($u(R) = 0$ and $v(R) > 0$); (middle) $v(0) = 11.4$ ($u(R) = v(R) = 0$, crossing solution); (right) $v(0) = 13$ ($u(R) > 0$ and $v(R) = 0$).

Numerical Methods

We use an numerical method to solve an initial value problem such as (1.7). We consider a more general problem:

$$\begin{aligned} &u_1'' + \frac{n-1}{r}u_1' + f(u_1,u_2) = 0,\ r > 0, \\ &u_2'' + \frac{n-1}{r}u_2' + g(u_1,u_2) = 0,\ r > 0, \\ &u_1(0) = A > 0,\ u_1'(0) = 0, \\ &u_2(0) = B > 0,\ u_2'(0) = 0, \end{aligned} \qquad (3.1)$$

Where f and g are appropriate nonlinear functions and A, B > 0. We first expand the system (3.1) from two second order differential equations to four first order differential equations:

$$\begin{aligned} u_1' &= v_1, \\ v_1' &= \frac{-n-1}{r}u_1' - f(u_1,u_2), \\ u_2' &= v_2, \\ v_2' &= \frac{-n-1}{r}u_2' - g(u_1,u_2), \\ u_1(0) &= A > 0,\ v_1(0) = 0, \\ u_2(0) &= B > 0,\ v_2(0) = 0. \end{aligned} \qquad (3.2)$$

First, we discretize the space of initial values $\{(A, B) : A_b \le A \le A_e, B_b \le B \le B_e\}$ to a two dimensional data structure:

$$\{(A_i, B_j) : 0 \le i \le n, 0 \le j \le m\},$$

where $A_i = A_b + (i/n)(A_e - A_b)$ and $B_j = B_b + (j/n)(B_e - B_b)$. Then for each initial value (A_i, B_j), we solve (3.2) by using an appropriate ODE solver in MatLab until the solution reaches a stopping time defined by $T = sup\{r>0 : u_1(r)v_1(r)u_2(r)v_2(r) \neq 0\}$. In fact, we only detect the stopping time if the initial value (A, B) is valid, which means that is satisfies f(A, B) > 0 and g(A, B) > 0. That is, if (A, B) belongs to region I defined in (2.5). If (A, B) ϵ $II \cup III \cup IV$, then initially u'(r) > 0 or v'(r)>0 for small R >0, and the solution cannot be the one we desire. On the bifurcation graph, we use the color "cyan" for the data point (A_i, B_j) if (A_i, B_j) ϵ $II \cup III \cup IV$. On the other hand, if the initial value (A_i, B_j) is an element of I, then for some δ > 0, $u_1(r)$, $u_2(r)$ > 0 and $u_1'(r)$, $u_2'(r)$ < 0 for r in (0, δ), hence T is well defined. As the solution reaches T, we color the data point according to the classifications:

$$\begin{aligned} B &= \{(A,B) \in I : T < \infty, u_1(T) = 0, u_1'(T) < 0, u_2(T) > 0, u_2'(T) < 0\}, \\ G &= \{(A,B) \in I : T < \infty, u_1(T) > 0, u_1'(T) = 0, u_2(T) > 0, u_2'(T) < 0\}, \\ R &= \{(A,B) \in I : T < \infty, u_1(T) > 0, u_1'(T) < 0, u_2(T) = 0, u_2'(T) < 0\}, \\ Y &= \{(A,B) \in I : T < \infty, u_1(T) > 0, u_1'(T) < 0, u_2(T) > 0, u_2'(T) = 0\}, \\ S &= \{(A,B) \in I : T < \infty, u_1(T) = 0, u_1'(T) < 0, u_2(T) = 0, u_2'(T) < 0\}, \\ Q &= \{(A,B) \in I : T = \infty, \lim_{r\to\infty} u_1(r) = \lim_{r\to\infty} u_2(r) = 0\}, \\ P &= I \setminus (B \cup G \cup R \cup Y \cup Q). \end{aligned} \qquad (3.3)$$

"Blue" for $u_1(T) = 0$, "green" for $u_1'(T) = 0$, "red for $u_2(T) = 0$, and "yellow" for $u_2'(T) = 0$.

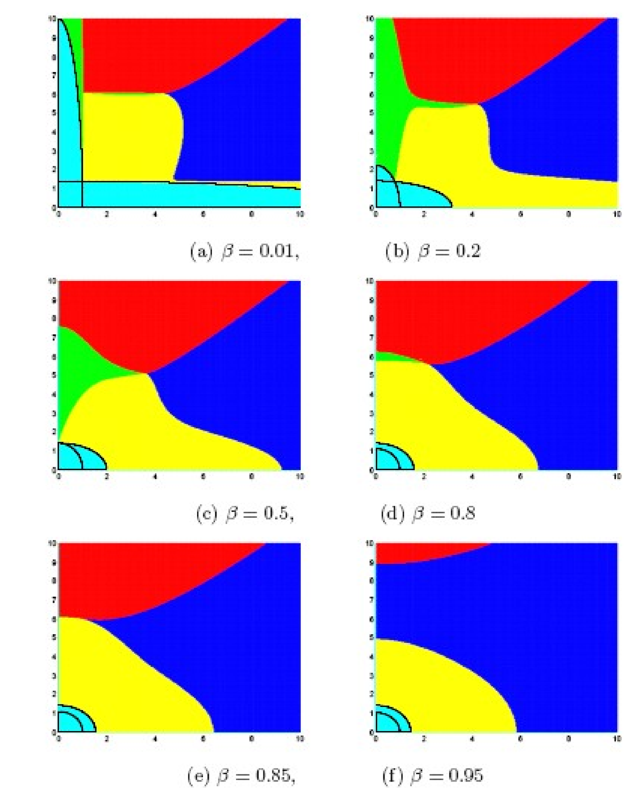

Figure 3: Bifurcation diagrams of (1.10). The coordinates are (A, B), the initial values in (3.1). Here $0 \le A, B \le 10$, 300×300 points in $(A, B) \in [0, 10]^2$ are sampled, $n = 3$, $\mu_1 = \mu_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$.

Numerical Bifurcation Diagrams

We investigate the qualitative behavior of solutions to the shooting problem (3.1). We used MatLab solver *ode113* since it handles computational intense models with an acceptable degree of accuracy.

Existence of bifurcation points can be shown from our numerical bifurcation diagrams of the shooting problem (3.1). In our numerical experiment, we fix a set of parameters (λ_1, λ_2, μ_1, μ_2) = (1, 2, 1, 1) and n = 3. We use β as a free parameter. In Fig. 3, one can see that $\beta_1^* \approx 0.85$. As $\beta \to (\beta_1^*)^-$, the green region shrinks to empty near (A, B) = (0, 6). This indicates a convergence of the ground states of the system to the semitrivial state ($u_1(r)$, $u_2(r)$).

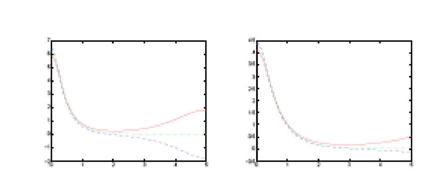

Figure 5: Ground state of (4.1) when $n = 3$. (left) $a = 2$, ground state $\phi_2(0) \approx 6.13$; (right) $a = 1$, ground state $\phi_1(0) \approx 4.32$.

Bibliography