Math 442 Lecture 2

How to solve a linear PDE ? $u(x,t)$

Separation of variables $u(x,t) = f(x)\,g(t)$

Example 7 $U_t = U_{xx}$ (diffusion equation)

$u(x,t) = f(x)g(t) \Rightarrow U_t = f(x)g'(t) \quad U_{xx} = f''(x)g(t)$

$$\frac{f(x)g'(t) = f''(x)g(t)}{f(x)\,g(t)} \Rightarrow \underbrace{\frac{g'(t)}{g(t)}}_{\substack{\text{not depend}\\ \text{on } x}} = \underbrace{\frac{f''(x)}{f(x)}}_{\substack{\text{not depend}\\ \text{on } t}} = -\lambda \Rightarrow \text{not depend on } x \text{ nor } t,$$

$-\lambda$ is a constant

$\Rightarrow$ a system of ODEs $\begin{cases} g'(t) = -\lambda g(t) \\ f''(x) = -\lambda f(x) \end{cases}$ (for the same λ)

$g(t) = e^{-\lambda t} \qquad f''(x) + \lambda f(x) = 0$.

Math 302 $f(x) = e^{kx} \Rightarrow f''(x) + \lambda f(x) = (k^2+\lambda)e^{kx} = 0 \Rightarrow k^2 = -\lambda$

$\Rightarrow k = \pm\sqrt{-\lambda} \Rightarrow f(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$

Case 1 $\lambda < 0$ $(-\lambda > 0)$ $u(x,t) = e^{-\lambda t}\left(c_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}\right)$

introduce $\sinh(x) = \dfrac{e^x - e^{-x}}{2} \quad \cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$$u(x,t) = e^{-\lambda t}\left(c_1 \sinh(\sqrt{-\lambda}x) + c_2\cosh(\sqrt{-\lambda}x)\right)$$

Case 2 $\lambda = 0$ $g(t) = 1$ $f(x) = C_1 + C_2 x$ $(f''(x) = 0)$

Case 3 $\lambda > 0$ $(-\lambda < 0)$ $u(x,t) = e^{-\lambda t}\left(c_1 \sin(\sqrt{\lambda}x) + c_2\cos(\sqrt{\lambda}x)\right)$

$\sqrt{-\lambda} = \sqrt{\lambda}\,i$
$-\sqrt{-\lambda} = -\sqrt{\lambda}\,i$
$e^{iy} = \cos y + i\sin y \qquad e^{-iy} = \cos y - i\sin y$

Why use $\sinh(x)$, $\cosh(x)$ instead of e^x, e^{-x}?

①	$\sinh(x)$ is odd function $\cosh(x)$ is even	$\sin(x)$ is odd $\cos(x)$ is even	e^x, e^{-x} is neither.
②	$(\sinh x)' = \cosh x$ $(\cosh x)' = \sinh x$	$(\sin(x))' = \cos x$ $(\cos x)' = -\sin x$	$(e^x)' = e^x$ $(e^{-x})' = -e^{-x}$
③	$e^x = \cosh x + \sinh x$ $e^{-x} = \cosh x - \sinh x$	$e^{ix} = \cos x + i\sin x$ $e^{-ix} = \cos x - i\sin x$	

Summary: the solutions of $u_t = u_{xx}$ can be

① $u(x,t) = e^{-\lambda t}\left(c_1 \sinh(\sqrt{-\lambda}x) + c_2\cosh(\sqrt{-\lambda}x)\right)$ when $\lambda<0$

② $u(x,t) = e^{-\lambda t}\left(c_1 \sin(\sqrt{\lambda}x) + c_2\cos(\sqrt{\lambda}x)\right)$ when $\lambda>0$

③ $u(x,t) = c_1 + c_2 \cancel{x}$ when $\lambda=0$

Note: a) This method only finds separable solutions, not all solutions.

b) This does not always work.

c) Such solution are not restricted by boundary conditions

d) The solutions in ① are not physically meaningful, as they are not bounded.

——— ② are physically meaningful, as they are bounded.

Alternatively (complex form): $u(x,t) = e^{-\lambda t}\left(c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}\right)$

If the linear PDE has constant coefficients, then another way is assume

$$u(x,t) = e^{\lambda t + kx} \quad (\text{for } \lambda, k \text{ possibly complex numbers})$$

$$u_t = \lambda u \quad u_{xx} = k^2 u \Rightarrow u_t = u_{xx} \Rightarrow \lambda = k^2$$

$\lambda = k^2$ is called dispersion relation of this PDE

Example 8 $\quad u_{xx} - xu_y + xu = 0$

$u(x,y) = f(x)g(y) \quad u_{xx} - xu_y + xu = f''(x)g(y) - xf(x)g'(y) + xf(x)g(y) = 0$

dividing by $xf(x)g(y)$ $\quad \dfrac{f''(x)}{xf(x)} - \dfrac{g'(y)}{g(y)} + 1 = 0$

Let $\dfrac{f''(x)}{xf(x)} = -\lambda$ then $\dfrac{g'(y)}{g(y)} = -\lambda + 1$

$$\begin{cases} f''(x) = -\lambda x f(x) \\ g'(y) = (-\lambda+1)g(y) \end{cases} \qquad g(y) = e^{(-\lambda+1)y}$$

$f''(x) + \lambda x f(x) = 0$ Airy's Equation

Not all separated linear ODE is solvable.

Now adding Boundary conditions

$$\begin{cases} u_t = u_{xx} & 0 < x < L, \\ u(0,t) = u(L,t) = 0 \end{cases}$$

Separating BC $\quad u(x,t) = f(x)g(t) \quad u(0,t) = f(0)g(t) = 0$.

If $g(t) = 0 \Rightarrow u(x,t) \equiv 0$ which just gives zero solution.

So $f(0) = 0$ similarly $f(L) = 0$

So $f(x)$ satisfies $\begin{cases} f''(x) + \lambda f(x) = 0 & 0 < x < L \\ f(0) = f(L) = 0 \end{cases}$

This is a boundary value problem of an ODE!

We have solved $f''(x) + \lambda f(x) = 0$ for all $\lambda \in \mathbb{R}$.

Now we select values of λ which can satisfy $f(0) = f(L) = 0$.

Case 1 $\quad \lambda < 0 \quad f(x) = C_1 \sinh(\sqrt{-\lambda}x) + C_2 \cosh(\sqrt{-\lambda}x) \qquad \sinh(0) = 0 \quad \cosh(0) = 1$

$f(0) = C_2 = 0 \qquad f(L) = C_1 \sinh(\sqrt{-\lambda}L) = 0 \qquad \sinh(x) > 0$ for $x > 0$

So $C_1 = 0 \Rightarrow C_1 = C_2 = 0 \Rightarrow f(x) = 0 \Rightarrow u(x,t) = 0$ (not useful)

Case 2 $\lambda = 0$.

$f(x) = C_1 + C_2 x$ $f(0) = C_1 = 0$ $f(L) = C_2 L = 0 \Rightarrow C_2 = 0$

Again $C_1 = C_2 = 0$

Case 3 $\lambda > 0$ $f(x) = C_1 \sin(\sqrt{\lambda} x) + C_2 \cos(\sqrt{\lambda} x)$ $\sin 0 = 0$ $\cos 0 = 1$

$\cancel{f(0) = C_1 \cdot 0 + C_2 = 0 \Rightarrow C_2 = 0}$

$\cancel{\cos(\sqrt{\lambda} L) = 0 \Rightarrow \sqrt{\lambda} L = \frac{\pi}{2} + k\pi = (k+\frac{1}{2})\pi}$

$f(0) = C_1 \cdot 0 + C_2 \cdot 1 = C_2 = 0 \Rightarrow C_2 = 0$ $k \in \mathbb{N}$

$f(L) = C_1 \sin(\sqrt{\lambda} L) = 0 \Rightarrow \sin(\sqrt{\lambda} L) = 0 \Rightarrow \sqrt{\lambda} L = k\pi$,

$\sqrt{\lambda} = \frac{k\pi}{L}$ $\lambda = \left(\frac{k\pi}{L}\right)^2$

Summary the boundary condition $f(0) = f(L) = 0$ selects values of λ: $\lambda_k = \left(\frac{k\pi}{L}\right)^2$, $k = 1, 2, 3, 4 \cdots$ ($k \in \mathbb{N}$)

These numbers are called eigenvalues, and corresponding eigenfunctions are $f_k(x) = \sin(\sqrt{\lambda_k} x) = \sin\left(\frac{k\pi}{L} x\right)$

They are the eigenvalues and eigenfunctions of eigenvalue problem

$$\begin{cases} f'' + \lambda f = 0, & 0 < x < L \\ f(0) = f(L) = 0 \end{cases}$$

Example 9 Find Eigenvalues of

$$\begin{cases} y'' + \lambda y = 0, & 0 < x < L, \\ y(0) = y(L) + y'(L) = 0 \end{cases}$$

Case 1 $\lambda < 0$,

$$y = C_1 \cosh(\sqrt{-\lambda}\, x) + C_2 \sinh(\sqrt{-\lambda}\, x)$$

$$y' = \sqrt{-\lambda}\,(C_1 \sinh(\sqrt{-\lambda}\, x) + C_2 \cosh(\sqrt{-\lambda}\, x))$$

$y(0) = 0 \Rightarrow C_1 = 0$

$y(L) + y'(L) = C_2 \sin(\sqrt{-\lambda}\, L) + C_2 \sqrt{-\lambda} \cosh(\sqrt{-\lambda}\, x) = 0$

$\Rightarrow C_2 \left(\sinh(\sqrt{-\lambda}\, L) + \cosh(\sqrt{-\lambda}\, L) \cdot \sqrt{-\lambda} \right) = 0$

But the function $f(x) = \sinh(x) + x\cosh(x) > 0$ for $x > 0$. $\Rightarrow C_2 = 0$

$\Rightarrow C_1 = C_2 = 0$

Case 2 $\lambda = 0$ $y = C_1 + C_2 x$, $y' = C_2$ $y(0) = 0 \Rightarrow C_1 = 0$ $y(L) + y'(L) = C_2 L + C_2 = 0 \Rightarrow C_2 = 0$

Case 3 $\lambda > 0$ $y = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$

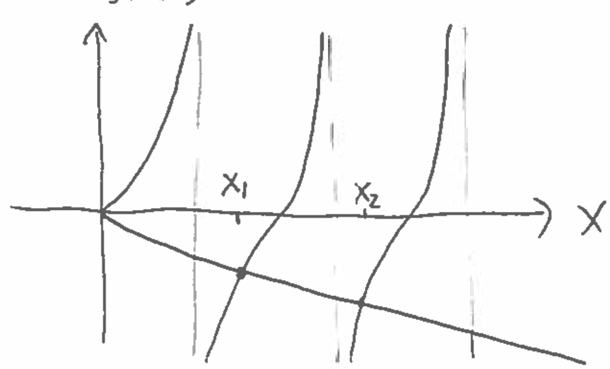
$$y' = \sqrt{\lambda}\left(-C_1 \sin(\sqrt{\lambda}\, x) + C_2 \cos(\sqrt{\lambda}\, x)\right)$$

$y(0) = 0 \Rightarrow C_1 = 0$ $y(L) + y'(L) = C_2 \sin(\sqrt{\lambda}\, L) + \sqrt{\lambda}\, C_2 \cos(\sqrt{\lambda}\, L) = 0$.

$\Rightarrow C_2 \left(\sin(\sqrt{\lambda}\, L) + \sqrt{\lambda} \cos(\sqrt{\lambda}\, L)\right) = 0 \Rightarrow \dfrac{\sin(\sqrt{\lambda} L)}{\cos(\sqrt{\lambda} L)} + \sqrt{\lambda} = 0$.

$\Rightarrow \tan(\sqrt{\lambda} L) = -\sqrt{\lambda}$

Let $f_1(x) = \tan(Lx)$ and $f_2(x) = -x$ for $x > 0$



Let the intersection points of $f_1(x)$ and $f_2(x)$ be $x_1, x_2 \cdots$ s.t $0 < x_1 < x_2 < \cdots$

Then $\lambda_n = x_n^2$ is the eigenvalue

eigenfunction $y_n(x) = \sin(\sqrt{\lambda_n}\, x)$

$= \sin(x_n x)$

$T'' + c^2 \lambda T = 0$

① $\lambda < 0$ $\quad T(t) = c_3 \sinh(c\sqrt{\lambda}t) + c_4 \cosh(c\sqrt{\lambda}t)$

② $\lambda = 0$ $\quad T(t) = C_3 + C_4 t$

③ $\lambda > 0$ $\quad T(t) = C_3 \cos(c\sqrt{\lambda}t) + C_4 \sin(c\sqrt{\lambda}t)$

$u_{tt} = c^2 u_{xx}$ $\qquad \lambda > 0$

separable solution: ① $u(x,t) = (c_1 \sinh\sqrt{\lambda}x + c_2 \cosh\sqrt{\lambda}x)(c_3 \sinh(c\sqrt{\lambda}t) + c_4 \cosh(c\sqrt{\lambda}t))$

<u>unbounded in t</u>

② $\lambda = 0$ $\quad u(x,t) = (C_1 + C_2 x)(C_3 + C_4 t)$ $\quad$ <u>unbounded in t</u>

③ $\lambda < 0$ $\quad u(x,t) = (C_1 \cos\sqrt{-\lambda}x + C_2 \sin\sqrt{-\lambda}x)(C_3 \cos(\sqrt{-\lambda}t) + C_4 \sin(c\sqrt{-\lambda}t))$

<u>bounded in t</u>

λ = frequency. or wave number.

Selection of frequency $\Rightarrow$ boundary condition!

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < l,\ t > 0. \\ \boxed{u(0,t) = u(l,t) = 0, \quad \rightarrow \text{Dirichlet BC}} \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) \rightarrow \text{Initial condition (Chap 5)} \end{cases}$$

Separation of variables $\quad u(x,t) = X(x)T(t)$

$X(0)T(t) = 0 \qquad X(l)T(t) = 0$

$\Rightarrow X(0) = 0, \quad X(l) = 0.$

So possible solutions of $\begin{cases} u_{tt} = c^2 u_{xx}, & 0<x<l \\ u(0,t) = u(l,t) = 0 \end{cases}$

are $u_k(x,t) = \cdot T_k(t) X_k(x)$

$$= \left(C_{3k}\cos\left(c\frac{k\pi}{l}t\right) + C_{4k}\sin\left(c\frac{k\pi}{l}t\right)\right)\cdot \sin\left(\frac{k\pi}{l}x\right)$$

linear Principle

$$u(x,t) = \sum_{k=1}^{\infty} \left(C_{3k}\cos\left(c\frac{k\pi}{l}t\right) + C_{4k}\sin\left(c\frac{k\pi}{l}t\right)\right)\sin\left(\frac{k\pi}{l}x\right)$$

This is the general solution of equation.

initial condition $\Rightarrow$ determine C_{3k}, C_{4k}. (Chap 5)

This is called <u>Fourier series solution</u>

vibrating frequency : $\frac{k\pi}{l}\cdot c = \frac{k\pi}{l}\cdot\sqrt{\frac{T}{\rho}}$ $\quad k = 1, 2, 3 \cdots$

(Euler 1749), (Fourier 1827).

<u>Diffusion Equation</u> $\begin{cases} u_t = k u_{xx}, & 0<x<l \\ u(0,t) = u(l,t) = 0. \end{cases}$

$u(x,t) = T(t)X(x)$ $\quad u_t = T'\cdot X \quad u_{xx} = T\cdot X''$

$$T'(t)\cdot X(x) = k\,T(t)\cdot X''(x) \Rightarrow \frac{T'(t)\cdot X(x)}{T(t)\cdot X(x)} = k\frac{T(t)X''(x)}{T(t)X(x)}$$

$$\Rightarrow \frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)} = -k\lambda \Rightarrow X''(x) = -\lambda X(x)$$

$$\begin{cases} X'' + \lambda X = 0 & 0<x<l \\ X(0) = X(l) = 0 \end{cases}$$

So same eigenvalue/eigenfunction

$$\lambda_k = \left(\frac{k\pi}{l}\right)^2 \qquad X_k(x) = \sin\left(\frac{k\pi}{l}x\right) \qquad k = 1, 2, 3, \cdots$$

$$T'(t) = -k\lambda_k T(t) \Rightarrow T(t) = e^{-k\lambda_k t} = e^{-k\left(\frac{k\pi}{l}\right)^2 t}$$

series solution

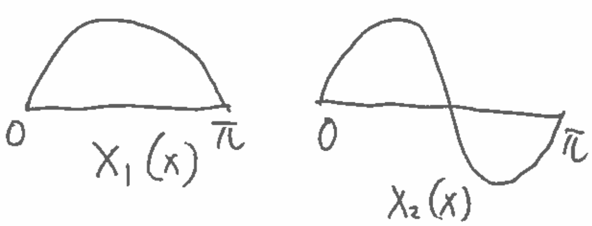
$$u(x,t) = \sum_{k=1}^{\infty} C_k e^{-k\left(\frac{k\pi}{l}\right)^2 t} \sin\left(\frac{k\pi}{l}x\right)$$

Homework problems.

Eigenvalue Problem

$$\begin{cases} \cancel{X}'' + \lambda X = 0, & 0<x<l \\ X(0) = X(l) = 0 \end{cases} \qquad \text{Dirichlet BC.}$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n = \sin\left(\frac{n\pi}{l}x\right)$$



So $X_n(x)$ has $(n-1)$ zeros in $(0,\pi)$

$$\mathcal{L}(X) = -X'', \quad \text{with } X(0) = X(l) = 0$$

is a linear operator with eigenvalues $\left(\frac{n\pi}{l}\right)^2$, $n = 1, 2 \cdots$

infinitely many eigenvalues

Compared to $Ax = \lambda x$ (linear algebra $n\times n$ matrix)

n eigenvalues.