Math 442 Lecture 8

linear PDE $\quad u_{tt} = c^2 u_{xx}, \quad u_t = k u_{xx}, \quad x \in \mathbb{R}$.

If $u(x,t)$ is a solution, so is (regardless of initial condition)

(1) $u(x-x_0, t-t_0)$ (translation in space or time)

(2) $u(2x_0 - x, t)$ (reflection in space)

(also $u(x, 2t_0 - t)$ for wave eq., not diffusion eq.)

(3) linear combinations of two solutions

(4) any derivatives u_x, u_t, u_{xx} of u

(5) dilation $\quad u(ax, at)$ for wave

$\qquad u(\sqrt{a}x, at)$ for diffusion

(6) $\int_{-\infty}^{\infty} g(y)\, u(x-y)\, dy$ (integral of translation in y)

<u>Proof</u> (2) Let $v(x,t) = u(2x_0 - x, t)$

$$V_{tt} = u_{tt}(2x_0-x,t) \quad V_x(x,t) = -u_x(2x_0-x,t)$$

$$V_{xx}(x,t) = u_{xx}(2x_0-x,t)$$

Since $u_{tt}(2x_0-x, t) = c^2 u_{xx}(2x_0-x,t)$

then $V_{tt}(x,t) = c^2 V_{xx}(x,t)$

(4) Differentiating $u_{tt} = c^2 u_{xx}$ in x, $u_{ttx} = c^2 u_{xxx}$

$$u_{ttx} = (u_x)_{tt} \qquad u_{xxx} = (u_x)_{xx}$$

So for $V = u_x$

$$V_{tt} = c^2 V_{xx}$$

Solve Diffusion Equation

$u_t = k u_{xx}$ not as simple as wave equation

Find special solution $u(x,t) = v\left(\frac{x^2}{t}\right)$

$u_t = v'\left(\frac{x^2}{t}\right)\cdot\left(-\frac{x^2}{t^2}\right)$ $\quad u_x = v'\left(\frac{x^2}{t}\right)\cdot\frac{2x}{t}$

$u_{xx} = v''\left(\frac{x^2}{t}\right)\cdot\frac{2x}{t}\cdot\frac{2x}{t} + v'\left(\frac{x^2}{t}\right)\cdot\frac{2}{t} = v''\left(\frac{x^2}{t}\right)\cdot\frac{4x^2}{t^2} + v'\left(\frac{x^2}{t}\right)\frac{2}{t}$

$\Rightarrow v'\cdot\left(-\frac{x^2}{t^2}\right) = k\left(v''\cdot\frac{4x^2}{t^2} + v'\cdot\frac{2}{t}\right)$ multiplying t

$\Rightarrow -v'\cdot\left(\frac{x^2}{t}\right) = k v''\frac{4x^2}{t} + kv'\cdot 2$ $\quad$ Let $s = \frac{x^2}{t}$

$\Rightarrow -s v'(s) = k v''(s)\cdot 4s + 2kv'(s)$

$\Rightarrow 4ks v''(s) = -(2k+s)v'(s)$ $\quad$ Let $w(s) = v'(s)$

$4ks w'(s) = -(2k+s)w(s)$

$$\frac{dw}{w} = -\frac{(2k+s)}{4ks} \Rightarrow \int\frac{dw}{w} = -\int\frac{(2k+s)}{4ks}ds$$

$\Rightarrow \ln w = -\int\frac{1}{2s}ds - \int\frac{1}{4k}ds \Rightarrow \ln w = -\frac{1}{2}\ln s - \frac{1}{4k}s + C$

$\Rightarrow \ln w = \ln s^{-\frac{1}{2}} - \frac{1}{4k}s + C \Rightarrow w = s^{-\frac{1}{2}}e^{-\frac{1}{4k}s}\cdot C$

So $v'(s) = w(s) = \frac{C}{\sqrt{s}}e^{-\frac{s}{4k}}$

$$v(s) = C\int\frac{1}{\sqrt{p}}e^{-\frac{p}{4k}}dp = C_1\int_0^s\frac{1}{\sqrt{p}}e^{-\frac{p}{4k}}dp + C_2$$

We find solution:

$$u(x,t) = C_1\int_0^{\frac{x^2}{t}}\frac{1}{\sqrt{p}}e^{-\frac{p}{4k}}dp + C_2$$

More solutions:

$$U_t(x,t) = -C_1 \frac{x}{t^{3/2}} e^{-\frac{x^2}{4kt}}, \quad U_x(x,t) = 2C_1 \frac{1}{t^{1/2}} e^{\frac{-x^2}{4kt}}$$

(solutions without integral !)

Particular one : $u(x,t) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4kt}}$

~~Integral of transition:~~

For fixed t, $u(x,t)$ looks like a probability distribution)
(normal distribution

In that case $\int_{-\infty}^{\infty} u(x,t)\,dx = 1$

$$\int_{-\infty}^{\infty} c \cdot \frac{1}{\sqrt{t}} e^{\frac{-x^2}{4kt}} dx \qquad \text{Let } y = \frac{x}{\sqrt{4kt}} \quad \text{then } dy = \frac{dx}{\sqrt{4kt}}$$

$$= \int_{-\infty}^{\infty} c\sqrt{4k}\, e^{-y^2} dy = c\sqrt{4k} \int_{-\infty}^{\infty} e^{-y^2} dy = c\sqrt{4k} \cdot \sqrt{\pi} = 1$$

$$\boxed{\text{important integral : } \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}}$$

So $C = \frac{1}{\sqrt{4k\pi}}$

Solution $u(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}}$ a normal distribution solution for diffusion equation !

mean $= 0$ variance $= 2kt$ standard deviation $= \sqrt{2kt}$

$$N(x,t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\begin{cases} u_t = k u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = \phi(x), & x \in \mathbb{R}. \end{cases}$$

Solution: $$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy$$

(integral of translations)

proof Let $$S(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}} \rightarrow \text{pdf (probability distribution function)}$$

$$Q(x,t) = \int_{-\infty}^{x} S(y,t)\, dy \rightarrow \text{cdf (cumulative distribution function)}$$

$$= \left(\int_{-\infty}^{0} + \int_{0}^{x} \right) = \frac{1}{2} + \int_0^x S(y,t)\, dy$$

$$= \frac{1}{2} + \int_0^x \frac{1}{\sqrt{4k\pi t}} e^{-\frac{y^2}{4kt}} dy \qquad z = \frac{y}{\sqrt{4kt}} \quad dz = \frac{dy}{\sqrt{4kt}}$$

$$= \frac{1}{2} + \int_0^{\frac{x}{\sqrt{4kt}}} \frac{1}{\sqrt{\pi}} e^{-z^2} dz = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-z^2} dz$$

Claim $$Q(x,0) = \lim_{t \to 0} Q(x,t) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \quad \text{(Heaviside function)}$$

If $x > 0$ $$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-z^2} dz \xrightarrow{t \to 0} \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} dz = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}$$

$$= 1$$

If $x < 0$ $$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-z^2} dz \xrightarrow{t \to 0} \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{-\infty} e^{-z^2} dz = \frac{1}{2} - \frac{1}{2} = 0$$

① $u(x,t)$ satisfies $u_t - k u_{xx} = 0$ since it is an integral of translations.

② $$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, \phi(y)\, dy = \int_{y=-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\, \phi(y)\, dy$$

$$= \int_{y=-\infty}^{y=\infty} -\frac{\partial Q}{\partial y}(x-y,t)\, \phi(y)\, dy$$

$$= -Q(x-y,t)\,\phi(y)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} Q(x-y,t)\,\phi'(y)\,dy$$

$$= 0 + \int_{-\infty}^{\infty} Q(x-y,t)\,\phi'(y)\,dy$$

Now $u(x,0) = \int_{-\infty}^{\infty} Q(x-y,0)\,\phi'(y)\,dy = \int_{-\infty}^{x} \phi'(y)\,dy = \phi(y)\Big|_{-\infty}^{x}$

$$= \phi(x)$$

Here we assume that $\phi(y) \to 0 \quad (y \to \pm\infty)$ $\blacksquare$

(1) $S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$ fundamental solution / Green's function / diffusion kernel, heat kernel of diffusion equation (heat equation)

(2) $u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\,dy$

$\phi(y) \to$ initial distribution

diffusion $\to \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}}$ goes to x

(3) $\lim_{t\to 0} Q(x,t) = H(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$ Heaviside function

$\lim_{t\to 0} S(x,t) = \lim_{t\to 0} \frac{\partial S}{\partial x}(x,t) = \delta(x)$ Dirac Delta function

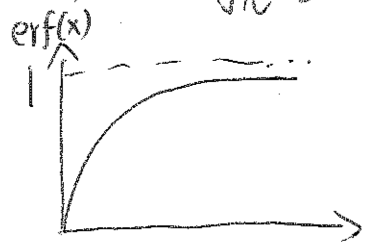
$$= H'(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

(4) $S(x,t)$ describes the diffusion of a point source, i.e. $\phi(x) = \delta(x)$ initially concentrates at $x=0$, then spread following diffusion equation, which is a normal distribution with variance $= 2kt$

Error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\,dp \qquad \operatorname{erf}(0) = 0 \qquad \lim_{x\to\infty} \operatorname{erf}(x) = 1$$



erf is an odd afunction

erfc(x) complementary efror function $= 1 - \operatorname{erf}(x)$

Example P52 #1 Find solution of $u_t = k u_{xx}$ with $\phi(x) = \begin{cases} 1 & |x| < l \\ 0 & |x| > l \end{cases}$

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-1}^{1} e^{-\frac{(x-y)^2}{4kt}}\,dy$$

$$p = \frac{y-x}{\sqrt{4kt}} \qquad dp = \frac{dy}{\sqrt{4kt}} \Rightarrow dy = \sqrt{4kt}\,dp \qquad \cancel{x = y - \sqrt{4kt}\,p}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{\frac{-1-x}{\sqrt{4kt}}}^{\frac{1-x}{\sqrt{4kt}}} e^{-p^2} \sqrt{4kt}\,dp = \frac{1}{\sqrt{\pi}} \int_{\frac{-1-x}{\sqrt{4kt}}}^{\frac{1-x}{\sqrt{4kt}}} e^{-p^2}\,dp$$

$$= \frac{1}{2}\left[\operatorname{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{-1-x}{\sqrt{4kt}}\right)\right]$$

HW: $\phi(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x > 0 \end{cases}$

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} e^{-y}\,dy = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\frac{(x-y)^2 + 4kty}{4kt}}$$

algebra: $\dfrac{(x-y)^2 + 4kty}{4kt} = \dfrac{(y - O_1)^2 \pm O_2}{4kt}$

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}}\, e^{\frac{O_2}{4kt}} \int_{?}^{\infty} e^{-p^2}\,dp$$

$\longrightarrow$ erf