# Math 442 Lecture 7



<u>Wave equation</u> $\rho u_{tt} = T u_{xx}$ or $u_{tt} = c^2 u_{xx}$

Energy: Kinetic energy $\frac{1}{2}mv^2 = \frac{1}{2}\int_{-\infty}^{\infty} \rho u_t^2 dx = KE(t)$

$$\frac{d\,KE(t)}{dt} = \frac{\rho}{2}\int_{-\infty}^{\infty} 2u_t(x,t)\,u_{tt}(x,t)\,dx = \int_{-\infty}^{\infty} u_t \cdot \rho u_{tt}\,dx$$

(from wave equation) $$= \int_{-\infty}^{\infty} u_t \cdot T u_{xx}\,dx = T\int_{-\infty}^{\infty} u_t\, du_x = T u_t u_x\Big|_{-\infty}^{\infty} - T\int_{-\infty}^{\infty} u_x u_{xt}\,dx$$

$$= -T\int_{-\infty}^{\infty} u_x u_{xt}\,dx$$

potential energy $PE(t) = \frac{1}{2}\int_{-\infty}^{\infty} T u_x^2(x,t)\,dx$

$$\frac{d\,PE(t)}{dt} = \frac{1}{2}T\int_{-\infty}^{\infty} 2u_x(x,t)\,u_{xt}(x,t)\,dx = T\int_{-\infty}^{\infty} u_x u_{xt}\,dx$$

So $$\frac{d\,KE(t)}{dt} = -\frac{d\,PE(t)}{dt}$$

Define total energy $E(t) = KE(t) + PE(t) = \frac{1}{2}\int_{-\infty}^{\infty}\left(\rho u_t^2(x,t) + T u_x^2(x,t)\right)dx$

Then $\frac{dE}{dt} = 0$ (total energy is conserved)

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<u>math recap</u> ① If $f(t)$ satisfies $f'(t)=0$, then $f(t) \equiv f(0)$, $\forall t>0$, $f(t)$ is a <u>conserved quantity</u>.

② $\int_a^b u v_x\,dx = \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v u_x\,dx$ (integral by parts)

# Uniqueness of solution to wave equation

$$\begin{cases} \rho u_{tt} = T u_{xx}, & x \in \mathbb{R},\ t>0 \\ u(x,0) = \phi(x), & x \in \mathbb{R}, \\ u_t(x,0) = \psi(x), & x \in \mathbb{R}. \end{cases} \quad (*)$$

D'Alembert Solution $\quad u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$

Since we derive the solution, it should be the only one.

Prove : it has only one solution.

Proof Suppose $u_1(x,t)$ and $u_2(x,t)$ are two solutions of $(*)$

Let $v(x,t) = u_1(x,t) - u_2(x,t)$

Then $\begin{cases} \rho v_{tt} = T v_{xx}, & x \in \mathbb{R},\ t>0 \\ v(x,0) = 0, & x \in \mathbb{R}, \\ v_t(x,0) = 0. \end{cases}$

its energy $\quad E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho v_t^2 + T u_x^2)\,dx$

$$E(0) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho \cdot 0^2 + T 0^2) = 0$$

Since energy is conserved, then $E(t) = 0$

$$\int_{-\infty}^{\infty} \left(\rho v_t^2(x,t) + T u_x^2(x,t)\right) dx = 0$$

Since $\rho v_t^2(x,t) + T v_x^2(x,t) \geq 0, \quad \forall x \in \mathbb{R},$

then $\rho v_t^2(x,t) + T v_x^2(x,t) = 0, \quad \forall x \in \mathbb{R}$

$\Rightarrow v_t(x,t) \equiv 0, \quad v_x(x,t) \equiv 0$

Since $v(x,0) = 0$ then $v(x,t) = 0 \quad \forall x \in \mathbb{R}, t>0$

$\Rightarrow u_1(x,t) = u_2(x,t)$ it is unique! ▥

# Diffusion Equation (Uniqueness)

$$(*)\begin{cases} u_t = k u_{xx}, & 0<x<l,\ t>0 \\ u(x,0)=\phi(x), & 0<x<l, \\ u(0,t)=g(t),\ u(l,t)=h(t), & t>0 \end{cases}$$

(inhomogeneous Dirichlet BC)

Prove: ~~If the~~ the solution of $(*)$ is unique, if it exists.

Proof. Suppose $u_1(x,t)$ and $u_2(x,t)$ are two solutions

Let $V(x,t)=u_1(x,t)-u_2(x,t)$. Then

$$\begin{cases} V_t = kV_{xx}, \\ V(x,0)=0, \\ V(0,t)=0,\ V(l,t)=0 \end{cases}$$

Let $E(t)=\frac{1}{2}\int_0^l V^2(x,t)\,dx$

Then $\frac{dE}{dt}=\frac{1}{2}\int_0^l 2V(x,t)V_t(x,t)\,dx = \int_0^l V(x,t)V_{xx}(x,t)\,dx$

$= \int_0^l V\,dV_x = VV_x\Big|_0^l - \int_0^l V_x\cdot V_x\,dx = -\int_0^l V_x^2\,dx \le 0,$

So $E(0)=\frac{1}{2}\int_0^l V^2(x,0)\,dx = 0$ and $E'(t)\le 0$.

Note that $E(t)=\frac{1}{2}\int_0^l V^2(x,t)\,dx \ge 0$

So $0\le E(t)=\int_0^t E'(s)\,ds + E(0) \le 0+0 \le 0$

If $0\le E(t)\le 0$ then $E(t)=0$, for $t>0$.

From vanishing theorem, $V^2(x,t)=0$, $\forall x\in(0,l)$, $t>0$,

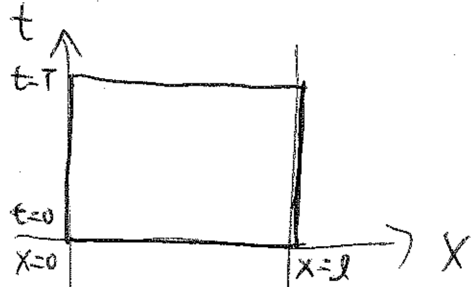
So $V(x,t)=0$ and $u_1(x,t)=u_2(x,t)$ $\forall x\in(0,l)$, $t>0$

The solution is unique. $\blacksquare$

# Maximum Principle

If $u(x,t)$ satisfies $u_t = k u_{xx}$ in a rectangle: $0 \le x \le l$, $0 \le t \le T$,

then the maximum value of $u(x,t)$ is achieved when $t=0$ or $x=0$ or $x=l$




① Max is on boundary, (but not $t=T$)

② Minimum also true ($-u$ also satisfies $u_t = k u_{xx}$)

~~③ If maximum is in interior, then~~

Is this true?

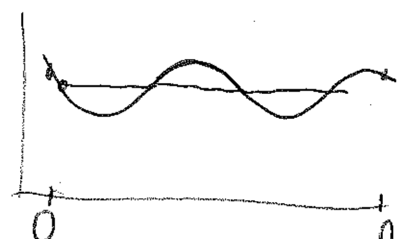
Example ① $u(x,t) = 0$ (max = 0, min = 0 achieve everywhere)

② $\begin{cases} u_t = u_{xx}, & 0 < x < \pi, \ t > 0 \\ u(x,0) = \sin x \end{cases}$

Solution $u(x,t) = e^{-t} \sin x$ $\quad u_t = -e^{-t} \sin x \quad u_{xx} = -e^{-t} \sin x$

max: $u(\frac{\pi}{2}, 0) = 1$ $(t=0)$, min: $u(0,t)=0$, $u(\pi,t)=0$ $\begin{pmatrix} x=0 \\ x=l \end{pmatrix}$

proof. If $u(x,t_1)$ is




$u_{xx} < 0$

then $u_t < 0$

$\Rightarrow u(x,t_2) < u(x,t_1)$

$\max u(x,t_2) < \max u(x,t_1)$

$u_{xx} > 0$

$u_t > 0$

$\Rightarrow u(x,t_2) > u(x,t_1)$

$\min u(x,t_2) > \min u(x,t_1)$

Diffusion = molecules move from higher density to lower density

$\Rightarrow$ more evenly distributed

$\Rightarrow$ max $\downarrow$ min $\uparrow$

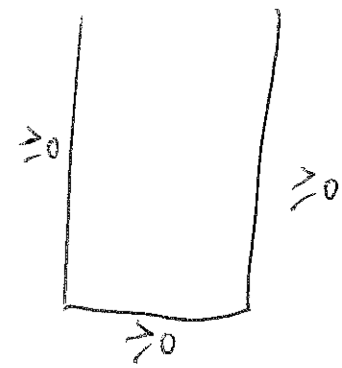
More rigorous proof $\Rightarrow$ textbook.

## Application of maximum principle

**Comparison Principle** if $u$ and $v$ are two solutions of $u_t = k u_{xx}$ for $0 \le x \le l$, and $u(x,0) \le v(x,0)$, $u(0,t) \le v(0,t)$, $u(l,t) \le v(l,t)$, then $u(x,t) \le v(x,t)$ for $0 \le t < \infty$, $0 \le x \le l$.

Proof. Let $w(x,t) = v(x,t) - u(x,t)$

$$\begin{cases} w_t = k w_{xx}, & 0 < x < l,\ t > 0 \\ w(x,0) = v(x,0) - u(x,0) \ge 0 \\ w(0,t) \ge 0, \quad w(l,t) \ge 0. \end{cases}$$




From Maximum Principle, the minimum of $w(x,t)$ is achieved on $t=0$, $x=0$ or $x=l$. So $\min w(x,t) \ge 0$. Thus $w(x,t) \ge 0$ for all $0 \le x \le l$, $t \ge 0$. ∎

**Uniqueness Again**

$$\begin{cases} u_t = k u_{xx} + f(x,t), & 0 < x < l,\ t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t), \quad u(l,t) = h(t) \end{cases}$$

Suppose $u_1(x,t)$, $u_2(x,t)$ are two solutions, Then $v(x,t) = u_1(x,t) - u_2(x,t)$ satisfies

$$\begin{cases} v_t = k v_{xx} \\ v(x,0) = 0 \\ v(0,t) = 0, \quad v(l,t) = 0 \end{cases}$$

$\max v(x,t) = 0$ and $\min v(x,t) = 0$

So $v(x,t) = 0$ for $0 \le x \le l$ and $t \ge 0$.