Math 442 Lecture 6

Chap 2 Wave, diffusion, when $x \in \mathbb{R}$ (no boundary)

Wave equation $u_{tt} = c^2 u_{xx}$, $-\infty < x < \infty$

$u_{tt} - c^2 u_{xx} = 0 \Rightarrow$ "Factoring the linear operators"

$u \longrightarrow \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial t} - c\frac{\partial}{\partial x}$ $\quad \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\cdot\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u(x,t) = 0$

$= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)(u_t + cu_x) = u_{tt} - cu_{xt} + cu_{tx} - c^2 u_{xx} = u_{tt} - c^2 u_{xx}$

So if $u_t + cu_x = 0$ then $u_{tt} - c^2 u_{xx} = 0$

if $u_t - cu_x = 0$ then $u_{tt} - c^2 u_{xx} = 0$

$u_t + cu_x = 0 \Rightarrow u(x,t) = f(x-ct)$

$u_t - cu_x = 0 \Rightarrow u(x,t) = g(x+ct)$

general solution $\boxed{u(x,t) = f(x+ct) + g(x-ct)}$

proof (by using coordinate method)

Define $\xi = x+ct$, $\eta = x-ct$

$u(x,t) = u(\xi(x,t), \eta(x,t))$ Then $u_x = u_\xi + u_\eta$, $u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_t = cu_\xi - cu_\eta$ and $u_{tt} = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}$

So $u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta} \Rightarrow u_{\xi\eta} = 0$

$u_\xi = \int u_{\xi\eta}\, d\eta = 0 + f(\xi) \Rightarrow u_\xi = f(\xi) \Rightarrow u = \int u_\xi\, d\xi = \int f(\xi)\, d\xi + g(\eta)$

$\Rightarrow u(\xi,\eta) = f(\xi) + g(\eta) \Rightarrow u(x,t) = f(x+ct) + g(x-ct)$

Initial-Value problem (Cauchy problem)

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x). \end{cases}$$

$u(x,t) = f(x+ct) + g(x-ct)$ Solve f and g

$u(x,0) = f(x) + g(x) = \phi(x)$

$u_t = c f'(x+ct) - c g'(x-ct)$

$u_t(x,0) = c f'(x) - c g'(x) = \psi(x)$

how to solve $\begin{cases} f + g = \phi(x) \\ c f' - c g' = \psi(x) \end{cases}$

$$\begin{aligned} f' + g' &= \phi'(x) \\ f' - g' &= \frac{\psi(x)}{c} \end{aligned} \Rightarrow f' = \frac{1}{2}\left(\phi' + \frac{\psi}{c}\right),\quad g' = \frac{1}{2}\left(\phi' - \frac{\psi}{c}\right)$$

$$\Rightarrow f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi(y)\,dy + A,\quad g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi(y)\,dy + B$$

So $u(x,0) = f(x) + g(x) = \phi(x) + A + B = \phi(x) \Rightarrow A + B = 0.$

$$u(x,t) = f(x+ct) + g(x-ct) = \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi(y)\,dy + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct}\psi(y)\,dy = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy$$

d'Alembert Formula (1746)

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy$$

Example 1 (HW3 #1)

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in \mathbb{R} \\ u(x,0) = e^x \\ u_t(x,0) = \sin x \end{cases}$$

$$u(x,t) = \frac{1}{2}\left(e^{x+ct} + e^{x-ct}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin y \, dy$$

$$= \frac{1}{2} e^x \left(e^{ct} + e^{-ct}\right) + \frac{1}{2c}\left(-\cos(x+ct) + \cos(x-ct)\right)$$

Solution : $e^x \cosh(ct) + \frac{1}{c} \sin x \sin ct$

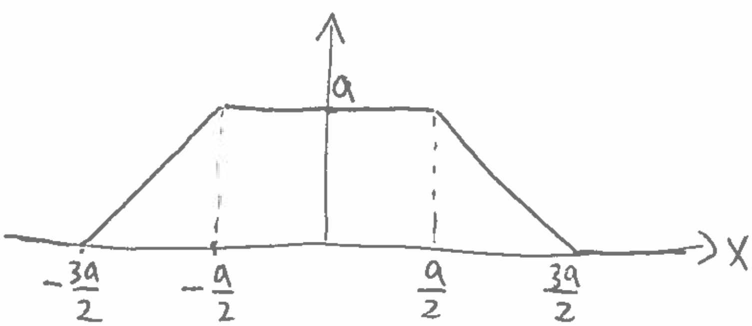
Example 2 P38 #5 "hammer"

$$\phi(x) \equiv 0, \quad \psi(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

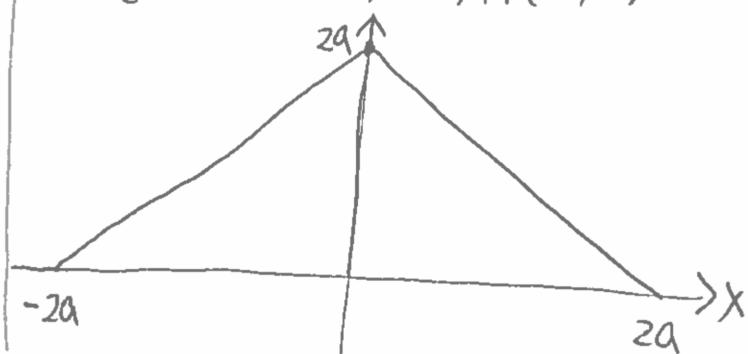
$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy = \frac{1}{2c}\{\text{length of } (x-ct, x+ct) \cap (-a,a)\}$$

A few snapshots

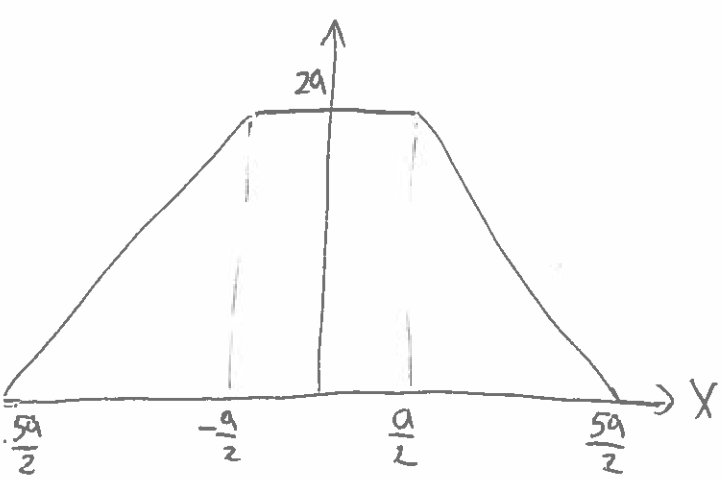
$t = \frac{a}{2c} \Rightarrow (x - \frac{a}{2}, x + \frac{a}{2}) \cap (-a,a)$



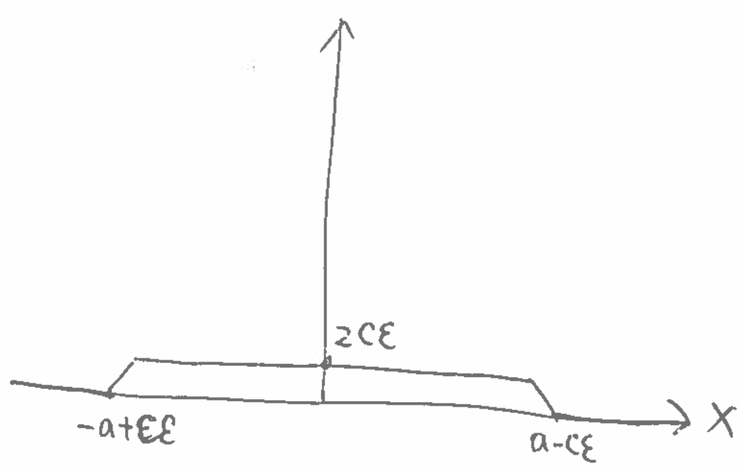
$t = \frac{a}{c} \Rightarrow (x-a, x+a) \cap (-a,a)$



$t = \frac{3a}{2c} \Rightarrow (x - \frac{3a}{2}, x + \frac{3a}{2}) \cap (-a,a)$



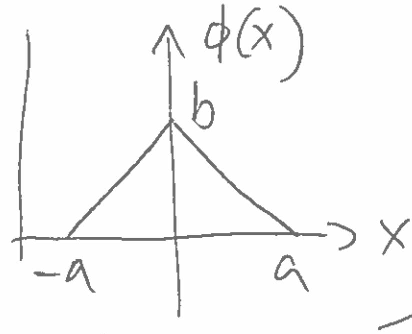
$t = \varepsilon \quad (x - c\varepsilon, x + c\varepsilon) \cap (-a,a)$



Plucked String

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in \mathbb{R}, \\ u(x,0) = \phi(x) \\ u_t(x,0) = 0 \end{cases}$$

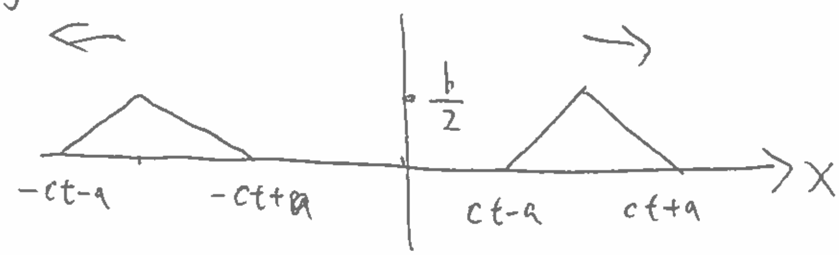
$$\phi(x) = \begin{cases} b\left(1 - \frac{|x|}{a}\right) & |x| < a \\ 0 & |x| > a \end{cases}$$



$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right]$$

split into 2 → left, → right

t large

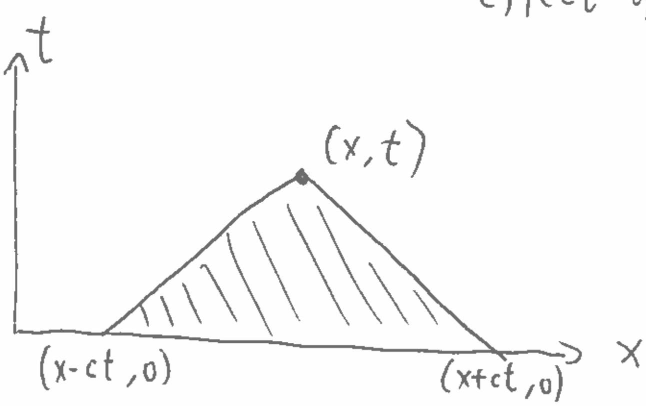


Causality

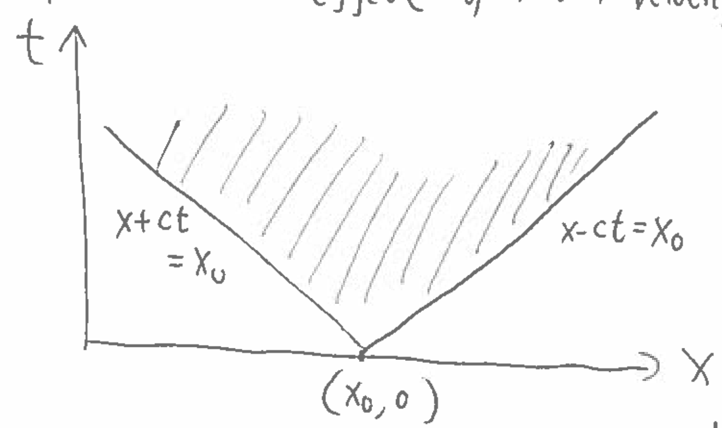
$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy$$

↓ effect of initial position

↓ effect of initial velocity



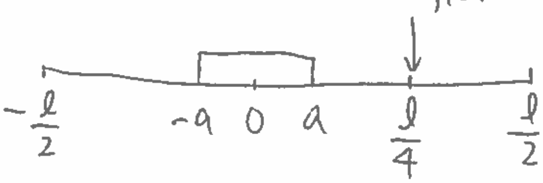
domain of dependence: $u(x,t)$ depends only on the initial values in $(x-ct, x+ct)$



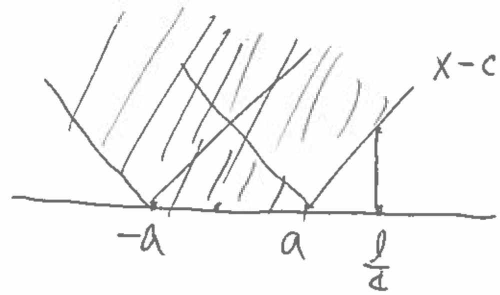
domain of influence: the initial condition at $x = x_0$ will affect the wave profile in the region

$$|x - x_0| \le ct$$

Example page 38 #3



Solve?



$x-ct=a$ and $\frac{l}{4}=x$

$\Rightarrow \; t=\,?$

Comparison of advection and wave equations

$$\begin{cases} u_t - c\,u_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

$$u(x,t) = \phi(x+ct)$$

$\downarrow$

wave moving to one direction only

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = 0 \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right]$$

$\downarrow$

wave splits in half, and moves to both left and right direction