Math 442 Lecture 5

Examples of BC:

A. Temperature on an object $D$: $Ut = D \Delta u$ (heat equation)
1. perfectly insulated: $\frac{\partial u}{\partial n} = 0$, $x \in \partial D$ (no flux BC)
2. If the exterior environment has temperature $T(t)$, then $u = T(t)$, $x \in \partial D$.
3. On the boundary, heat is exchanges between $D$ and outside, and it satisfies Newton's cooling law $\frac{\partial u}{\partial n} = -\alpha (u(x,t) - \Theta T(t))$ (Robin)

B. Jurassic Park. $Ut = D \Delta u + f(t, x, u)$ $u =$ density of dinosaurs
1. tall wall: $\frac{\partial u}{\partial n} = 0$, $x \in \partial D$, (reflective BC)
2. electric wall: $u = 0$, $x \in \partial D$ (lethal boundary condition)

C. Wild life reserve $Ut = D \Delta u + f(t, x, u)$
1. boundary = fence, $\frac{\partial u}{\partial n} = 0$ (Neumann)
2. boundary = highway without fence, $u = 0$ (Dirichlet)
3. boundary = fence with $\alpha$

D. Chemical in river $Ut = -D \Delta u - C$
$Ut = \frac{DU_{xx} - CU_x - KU}{\text{diffusion advection decay}}$
$x = 0$ upstream
$x = L$ downstream

1. $DU_x - CU = 0$, $x = 0, L$ (no flux) closed river?
2. $DU_x(0, t) - CU(0, t) = 0$, $DU_x(b, t) - CU(b, t) = K_2 u(b, t)$ (wash out rate)
$K_1, K_2$ positive or negative?

\[ \begin{align*}
0 & \quad \text{normal} = -1 \\
L & \quad \text{normal} = 1
\end{align*} \]

\[ (D {u_x}(0,t) - c {u}(0,t)) \cdot \text{(-1)} > 0. \]

outer normal flux

\[ (D {u_x}(L,t) - c {u}(L,t)) \cdot 1 \leq 0. \]

\[ \frac{\text{in flux}}{\text{dam}} = \frac{|J \cdot n|}{J} \]

Examples of steady state solution

\[ \begin{align*}
\text{(mixing problem)} & \\
\int U_t = D U_{xx}, & \quad 0 < x < L \\
\downarrow & \\
DU_x(0,t) = b - 1, & \quad DU_x(L,t) = b 0.2 U(L,t)
\end{align*} \]

input

wash out rate

steady state: \[ U'' = 0 \Rightarrow u(x) = ax + b, \quad u'(x) = a \]

\[ D a = -1 \Rightarrow a = -\frac{1}{D} \]

\[ D \cdot (-\frac{1}{D}) = -0.2 (a \cdot L + b) \]

\[ \Rightarrow -\frac{1}{D} L + b = 5 \Rightarrow b = 5 + \frac{1}{D} L \]

\[ \Rightarrow u(x) = -\frac{1}{D} x + \frac{1}{D} L + 5 = \frac{1}{D} (L - x) + 5 \]

Problem 3-6 in HW

P25 #3

\[ \begin{align*}
\int U_t = D \Delta u, & \\
\int u(x,0) = f(x) & \Rightarrow \int \frac{\Delta u}{\Delta n} = 0 \\
\frac{\partial u}{\partial n} = 0 & \\
\int_D \Delta u = D \int_D \frac{\partial u}{\partial n} = 0.
\end{align*} \]
P25 #4. \( f(x) = \begin{cases} \ 0 & x < \frac{l}{2} \\ H & \frac{l}{2} < x < l \end{cases} \)

\[
\begin{align*}
\begin{cases}
  u'' + f(x) = 0 & 0 < x < l \\
  u(0) = u(l) = 0 & \\
  u''(0) = u(l) = 0 & \\
\end{cases}
\end{align*}
\]

(Continuous at \( \frac{l}{2} \))

 glued at \( x = \frac{l}{2} \)