Math 442 Lecture 4

Multi-Var. Calc

function $u(x,y,z)$, $\nabla u = (u_x, u_y, u_z)$ gradient

$u: \mathbb{R}^3 \to \mathbb{R}$ $\qquad \nabla u: \mathbb{R}^3 \to \mathbb{R}^3$ (vector field)

For a vector field $\vec{F}(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\qquad \operatorname{div} \vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

divergence

$\operatorname{div} \vec{F}: \mathbb{R}^3 \to \mathbb{R}$ $\qquad$ some people use $\nabla \cdot \vec{F} = \operatorname{div} \vec{F}$

Divergence Theorem Let D be a bounded domain in $\mathbb{R}^3$ with "good boundary" S. Let $\vec{n}$ be the outer normal vector on S. Let $\vec{F}$ be a C^1 vector field on $\bar{D} = D \cup S$. Then

$$\underbrace{\iiint_D \operatorname{div} \vec{F}\, dx\,dy\,dz}_{\text{3D integral}} = \underbrace{\iint_{\partial S} \vec{F} \cdot \vec{n}\, dS}_{\text{2D surface integral}}$$

$\vec{F} \cdot \vec{n}$ dot product for two vectors

Example (page 19 #9) $\vec{F} = r^2 \vec{x} = (x^2+y^2+z^2)(x,y,z)$

$$= (r^2 x, r^2 y, r^2 z)$$

D = ball with radius a and center $(0,0,0)$

$$\text{Left} = \iiint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\,dy\,dz = \iiint_D \left(r^2 + 2x^2 + r^2 + 2y^2 + r^2 + 2z^2 \right)$$

$$= \iiint_D 3r^2\, dx\,dy\,dz = \int_0^a \int_0^{2\pi} \int_0^{\pi} 3r^2 \cdot r^2 \sin\theta\, d\theta\, d\varphi\, dr = 4a^5\pi$$

$$\text{Right} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S r^2\vec{x}\cdot\frac{\vec{x}}{r} = \iint_S r\cdot\vec{x}\cdot\vec{x} = \iint_S r|\vec{x}|^2$$

$$= \iint_S r^3 = a^3\iint_S 1\,ds = a^3\cdot\text{Surface area of } S = a^3\cdot 4\pi a^2 = 4\pi a^5.$$

So $\iiint_D \operatorname{div}\vec{F}\,dxdydz = \iint_S \vec{F}\cdot\vec{n}\,dS$.

Diffusion Equation in 3D

$\vec{x} = (x, y, z)$

$u(\vec{x}, t)$ = density of a substance at location $\vec{x}$ and time t.

Take a region D, with boundary bdy $D = \partial D$

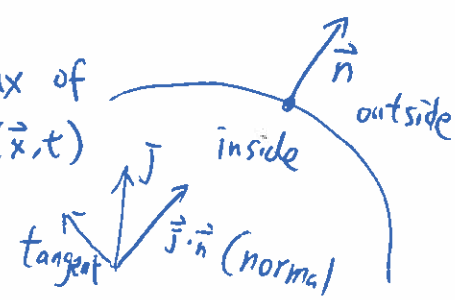
total mass in $D = \int_D u(\vec{x}, t)\,d\vec{x}$

$$\frac{d}{dt}\int_D u(\vec{x},t)\,d\vec{x} = \text{rate of immigration/emigration on } \partial D$$

$$= -\int_{\partial D} \vec{J}(\vec{x},t)\cdot\vec{n}\,dS$$

$\vec{J}$ = immigration rate at $(\vec{x}, t)$ = flux of $u(\vec{x},t)$

$\vec{n}$ = outer normal vector



Applying divergence theorem:

$$\int_{\partial D} \vec{J}\cdot\vec{n}\,ds = \int_D \operatorname{div}(\vec{J})\,d\vec{x}.$$

$$\Rightarrow \int_D u_t(\vec{x},t)\,d\vec{x} = -\int_D \operatorname{div}(\vec{J}(\vec{x},t))\,d\vec{x}$$

$$\int_D \left[u_t(\vec{x},t) + \operatorname{div}(\vec{J}(\vec{x},t))\right] d\vec{x} = 0.$$

Since D is an arbitrary region, then from the 2nd vanishing theorem,

$$u_t(\vec{x},t) + \operatorname{div}(\vec{J}(\vec{x},t)) = 0.$$

Conservation Law Equation

Assumption 1 (transport)

$$\vec{J}(\vec{x},t) = u(\vec{x},t)\,\vec{v}(\vec{x},t)$$

$\vec{v}(\vec{x},t)$ is a velocity field (river flow, ocean flow ... air flow)

$$u_t + \operatorname{div}(\vec{v}\cdot u) = 0 \quad \text{advection equation}$$

Assumption 2 (diffusion)

$$\vec{J}(\vec{x},t) = -D\nabla u(\vec{x},t) \quad (\text{here } \nabla \text{ is in } \vec{x} \text{ only})$$

Fick's law (for molecular movement) Fourier's law (for heat conduction)

$$u_t - \operatorname{div}(D\nabla u) = 0$$

$$u_t - D\operatorname{div}(\nabla u) = 0$$

$$\nabla u = (u_x, u_y, u_z) \qquad \operatorname{div}(\nabla u) = u_{xx} + u_{yy} + u_{zz} = \Delta u$$

Laplacian of u $\quad u_t - D\nabla^2 u = 0$

$$u_t - D\Delta u = 0$$

$$u_t - D\nabla\cdot\nabla u = 0$$

More general model

① $\frac{d}{dt}\int_D u(\vec{x},t)\,d\vec{x}$ = rate of immigration on ∂D + rate of birth/death/reaction in D

$f(t, \vec{x}, u)$

② $\vec{J}(\vec{x},t)$ = advection + diffusion

$$u_t = \underbrace{D\,\Delta u}_{\text{diffusion}} - \underbrace{c\,\vec{v}\cdot\nabla u}_{\text{advection}} + \underbrace{f(t,\vec{x},u)}_{\text{reaction}}$$

describing movement equation (passive diffusion active advection)

advection reaction diffusion equation

$$= D\left(u_{xx}+u_{yy}+u_{zz}\right) - c\left(v_1 u_x + v_2 u_y + v_3 u_z\right) + f$$

Steady state solution: solution $u(\vec{x},t) = u(\vec{x})$ (not depend on t)

$$u_t = 0$$

$$D\,\Delta u - c\,\vec{v}\cdot\nabla u + f(t,\vec{x},u) = 0$$

Corresponding kinetic equation (no diffusion, no advection)

$$u_t = f(t,u)$$ an ODE.