# Math 442 Lecture 3

## Application of 1st order linear PDE

(2) Age-structure population model

$$\frac{du}{dt} = bu \quad (\text{Malthus model}) \qquad u(t) = \text{total population}$$

idea: divide the population according to age

$$u(n) = \begin{pmatrix} u_1(n) \\ u_2(n) \\ \vdots \\ u_k(n) \end{pmatrix} \qquad u_i(n) = \text{population of the } i\text{-th age group at time } n$$

$$L = \begin{pmatrix} s_1 m_1 & s_1 m_2 & \cdots & s_1 m_{k-1} & s_1 m_k \\ s_2 & 0 & \cdots & 0 & 0 \\ 0 & s_3 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & s_k & 0 \end{pmatrix} \quad \text{Leslie matrix model (1945)}$$

$$u(n+1) = L \cdot u(n)$$

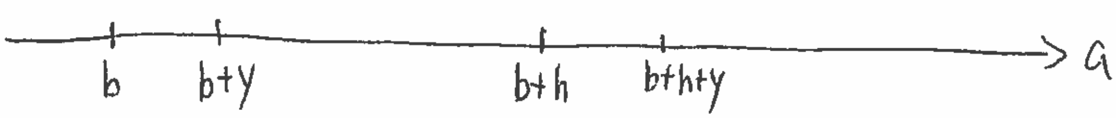
$s_i$ = survival rate from age $i-1$ to $i$ $\quad (0 < s_i < 1)$

$m_i$ = reproduction rate per capita at age $i$ $\quad m_i > 0$ for $\alpha \le i \le \beta$ fertile period

(Math 345) discrete time, discrete age structure

New idea: continuous time, continuous age structure

$u(t, a)$ = population density for age $a$ at time $t$




$$\underbrace{\int_b^{b+y} u(t,a)\,da}_{\substack{\text{population between} \\ \text{age } b \text{ and } b+y \\ \text{at time } t}} = \underbrace{\int_{b+h}^{b+h+y} u(t+h,a)\,da}_{\substack{\text{population between} \\ \text{age } b+h \text{ and } b+h+y \\ \text{at time } t+h}} \quad (\text{if no death!})$$

$$\int_b^{b+y} u(t,a)\,da = \int_{b+h}^{b+h+y} u(t+h,a)\,da$$

$\frac{d}{dy}$ $\qquad u(t, b+y) = u(t+h, b+h+y)$

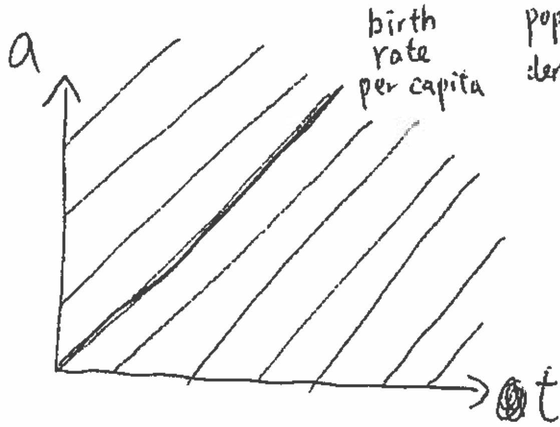
$\frac{d}{dh}$ $\qquad 0 = u_t(t+h, b+h+y)\cdot 1 + u_a(t+h, b+h+y)\cdot 1$

$y=0$ and $h=0$ $\qquad 0 = u_t(t,b) + u_a(t,b) = 0$

change $b \to a$ $\qquad 0 = u_t(t,a) + u_a(t,a)$

Equation of aging

$$\begin{cases} u_t + u_a = 0, & t>0,\ a>0, \\ u(0,a) = f(a), & a>0 \quad \text{(initial distribution)} \\ u(t,0) = \int_0^\infty \underbrace{b(a)}_{\text{birth rate per capita}} \underbrace{u(t,a)}_{\text{population density}}\,da = \text{total new birth} & \text{(boundary condition)} \end{cases}$$




$a > t \Rightarrow$ determined by initial condition

$t > a \Rightarrow$ determined by "new" birth

McKendrick – Von Foerster Equation

Question: If for age $a$, the death rate is $d(a)$, then what will be the equation?

$$\int_b^{b+y} u(t,a)\,da = \int_{b+h}^{b+h+y} u(t+h,a)\,(1+d(a))\,da$$

## Review of Multi-Var Calc (3)

$z = z(x, y)$ $\quad x = x(s,t)$, $\quad y = y(s,t)$

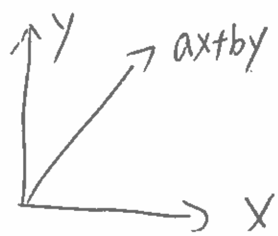
$z = z(x(s,t), y(s,t))$

chain rule:

$$z_s = z_x \cdot x_s + z_y \cdot y_s$$
$$z_t = z_x \cdot x_t + z_y \cdot y_t$$

---

## Change coordinates in a PDE




$$a u_x + b u_y + c u = 0$$

$ax+by$ and its orthogonal direction would be a good coordinate system!

$w = ax+by$ $\qquad$ new solution $u(w, z)$

$z = -bx+ay$ $\qquad = u(w(x,y), z(x,y))$

$$\begin{cases} u_x = u_w w_x + u_z z_x = a u_w - b u_z \\ u_y = u_w w_y + u_z z_y = b u_w + a u_z \end{cases}$$

So $a u_x + b u_y + c u = a(a u_w - b u_z) + b(b u_w + a u_z) + cu$

$$= (a^2+b^2) u_w + cu = 0$$

$$\Rightarrow u_w + \frac{c}{a^2+b^2} u = 0 \Rightarrow \frac{du}{dw} + \frac{c}{a^2+b^2} u = 0$$

$$\Rightarrow \frac{du}{u} = -\frac{c}{a^2+b^2} dw \Rightarrow \ln u = -\frac{c}{a^2+b^2} w + f(z)$$

$$\Rightarrow u = e^{-\frac{c}{a^2+b^2} w} e^{f(z)} \Rightarrow u(w,z) = e^{-\frac{c}{a^2+b^2} w} f(z)$$

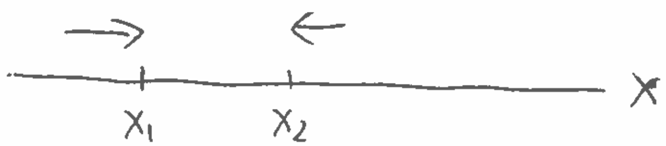
$$\Rightarrow u(x,y) = e^{-\frac{c}{a^2+b^2}(ax+by)} f(-bx+ay)$$

(when $c=0$ formula last time)

# Heat Equation (Diffusion Equation)

**Derivation 1** $\quad x \in (-\infty, \infty) \to$ river, $\quad t \to$ time

$u(x,t)$ = density of a pollutant in the river.




amount of pollutant in $[x_1, x_2]$

$$= \int_{x_1}^{x_2} u(x,t)\,dx = M(t)$$

$$\frac{dM}{dt} = \frac{d}{dt}\left(\int_{x_1}^{x_2} u(x,t)\,dx\right) = \int_{x_1}^{x_2} \frac{\partial u}{\partial t}(x,t)\,dx = \text{pollutant coming in from the boundary}$$

$= \text{flow to the right (?) at } x_1 + \text{flow to the left at } x_2$

**Assumption 1** flow rate is proportional to river velocity and pollutant density

flow to the right $= k_1 \cdot cu \quad$ ($c$ is the river velocity)

$$\int_{x_1}^{x_2} u_t(x,t)\,dx = k_1 c u(x_1,t) - k_1 c u(x_2,t) = k_1 c\left[u(x_1,t) - u(x_2,t)\right]$$

$$= k_1 c \int_{x_2}^{x_1} u_x(x,t)\,dx = -k_1 c \int_{x_1}^{x_2} u_x(x,t)\,dt$$

$$\Rightarrow \int_{x_1}^{x_2} \left(u_t(x,t) + k_1 c u_x(x,t)\right) dx = 0.$$

Since $x_1, x_2$ are arbitrary, then $u_t(x,t) + k_1 c\, u_x(x,t) = 0$ for all $t>0$, $x \in (-\infty, \infty)$

$$u_t + k_1 c\, u_x = 0 \quad \Rightarrow \quad \underline{\text{transport equation}}, \quad \underline{\text{advection equation}}$$

<u>Assumption 2</u> velocity $c = 0$ (river is not moving, or it is a 1D lake)

flow rate is proportional to $-u_x$ $= -k_2 u_x$
(flow direction is from high density to low density)

$$\int_{x_1}^{x_2} u_t(x,t)\,dx = k_2\left(-u_x(x_1,t)\right) - k_2\left(-u_x(x_2,t)\right)$$
$$= k_2\left(u_x(x_2,t) - u_x(x_1,t)\right)$$
$$= k_2 \int_{x_1}^{x_2} u_{xx}(x,t)\,dx$$

$$\Rightarrow \int_{x_1}^{x_2}\left(u_t(x,t) - k_2 u_{xx}(x,t)\right)dx = 0 \Rightarrow u_t - k_2 u_{xx} = 0 \Rightarrow \underline{\text{diffusion equation}}$$

<u>Assumption 3</u> = Assumption 1 + 2

$$u_t + k_1 c\, u_x - k_2 u_{xx} = 0 \quad \text{1-D } \underline{\text{diffusion-advection equation}}$$

<u>Mathematics Results used in derivation</u> (Appendix: Page 420, 416) ~~(Page 558, Theorem B.9, B.10)~~

(1) $\dfrac{d}{dt}\displaystyle\int_{x_1}^{x_2} u(x,t)\,dx = \int_{x_1}^{x_2} \frac{\partial u}{\partial t}(x,t)\,dx$

(2) If $\displaystyle\int_{x_1}^{x_2} f(x)\,dx = 0$ for any $x_1, x_2$ and $f(x)$ is continuous, then $f(x) = 0$ for all $x$.
(vanishing theorem)

<u>Assumption 2</u> = Fick's law (for molecule diffusion)
= Fourier's law (for heat conduction)

What diffuses? Almost everything (organisms, knowledge, technology, heat ...)
spillover, small mutation ...

An alternative approach is to assume
flow to the right $= \Phi(x,t)$

Then $\int_{x_1}^{x_2} u_t(x,t)\,dx = \Phi(x_1,t) - \Phi(x_2,t) = -\int_{x_1}^{x_2} \Phi_x(x,t)\,dx$

$\Rightarrow \quad u_t = -\Phi_x$ (equation of continuity) $\Rightarrow$ conservation law

assumption 1 $\quad \Phi(x,t) = k_1 \cdot c \cdot u(x,t)$

assumption 2 $\quad \Phi(x,t) = -k_2 u_x(x,t)$

More complicated: (1) $\Phi(x,t) = k_1 c(x,t) u(x,t) \Rightarrow u_t = -k_1 (cu)_x$

Fluid flow equation

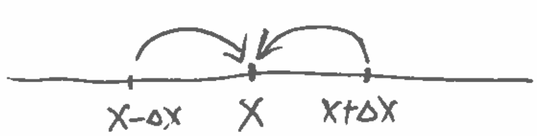
(2) $\Phi(x,t) = -k_2(x) u_x(x,t) \Rightarrow u_t = (k_2(x) u_x)_x$

spatially heterogeneous diffusion

Derivation 2 (random walk)

assume there are many particles move along x-axis, and $u(x,t)$ is the density of particles. Suppose for one particle at $x$, the probability that it "jumps" to $x-\Delta x$ or $x+\Delta x$ is $\frac{1}{2}$ each, then

$$u(x, t+\Delta t) = \frac{1}{2} u(x-\Delta x, t) + \frac{1}{2} u(x+\Delta x, t)$$




$\Delta t$ is the time needed for a "jump"

We use Taylor expansion: $f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2} f''(x)(\Delta x)^2 + \cdots$

$$u(x, t+\Delta t) = u(x,t) + u_t(x,t)\cdot \Delta t + \frac{1}{2} u_{tt}(x,t)(\Delta t)^2 + \cdots,$$

$$u(x\pm\Delta x, t) = u(x,t) \pm u_x(x,t)\,\Delta x + \frac{1}{2} u_{xx}(x,t)\cdot(\Delta x)^2$$

$$\Rightarrow \quad u_t(x,t)\cdot \Delta t + \frac{1}{2} u_{tt}(x,t)(\Delta t)^2 = \frac{1}{2} u_{xx}(x,t)(\Delta x)^2 + \cdots$$

$$\Rightarrow \quad U_t(x,t) + \frac{1}{2} U_{tt}(x,t)\,\Delta t = \frac{1}{2} U_{xx}(x,t) \frac{(\Delta x)^2}{\Delta t}$$

Ignoring higher order terms in Taylor expansion.

Let $\Delta t \to 0$, $\Delta x \to 0$ but $\frac{(\Delta x)^2}{\Delta t} \to D > 0$.

$$\Rightarrow \quad U_t = \frac{D}{2} U_{xx} \qquad \text{(diffusion equation)}$$

Assumption used:

① probability going to left and right are same

② Higher order terms in Taylor expansion can be ignored

③ The scale of spatial grid $\Delta x$ and time step $\Delta t$ satisfy $\frac{(\Delta x)^2}{\Delta t} = D$

if ① is not true then we get advection equation $U_t = -cU_x$ (more right)
$U_t = cU_x$ (more left)

So $U_t = DU_{xx}$ describes the movement which is not biased toward left or right.

If ② is not true. Telegraph Equation

$$AU_t + KU_{tt} = DU_{xx}$$

Diffusion Type Equation

Black-Scholes Equation

$$\frac{\sigma^2 S^2}{2} V_{SS} + rS V_S - rV + V_t = 0$$

$\Rightarrow$ can be changed to $U_\tau = \frac{\sigma^2}{2} U_{xx}$

$u_t$ = velocity (rate of change)

$u_{tt}$ = acceleration  Newton's 2nd law $mu_{tt} = F$. $\Rightarrow mu_{tt} + ku = 0$

Hooke's law $F = -ku$ ($u$ = distance)

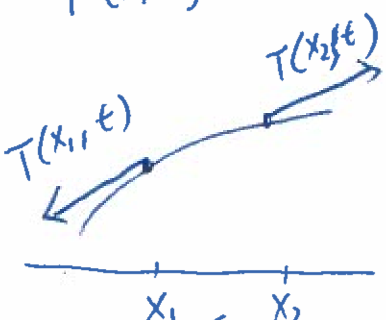
ODE: $mu'' + ku = 0$ (harmonic oscillator)

Vibrating String


each point moves vertically

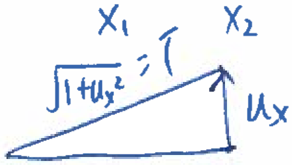

$u(x,t)$ = displacement from equilibrium position


$T(x,t)$ = tension vector, $\rho$ = density of the string




horizontal: $\left.\frac{T}{\sqrt{1+u_x^2}}\right|_{x_1}^{x_2} = 0$ (motion is vertical)

vertical: $\left.\frac{T u_x}{\sqrt{1+u_x^2}}\right|_{x_1}^{x_2} = \underbrace{\int_{x_1}^{x_2} \rho u_{tt}\, dx}_{\text{Newton's 2nd law}}$




$\boxed{\text{If } u_x \text{ is small,}}$ then $\sqrt{1+u_x^2} \approx 1$

$$T u_x \Big|_{x_1}^{x_2} = T\int_{x_1}^{x_2} u_{xx}\, dx \Rightarrow T\int_{x_1}^{x_2} u_{xx}\, dx = \int_{x_1}^{x_2} \rho u_{tt}\, dx$$

$$\Rightarrow \rho u_{tt} = T u_{xx} \Rightarrow u_{tt} = c^2 u_{xx} \quad c = \sqrt{\frac{T}{\rho}}$$ wave equation

If do not assume $u_x$ is small,

$$\rho_{tt} = \left(\frac{T u_x}{\sqrt{1+u_x^2}}\right)_x$$ nonlinear wave equation