Fourier transform and derivative

If $\mathcal{F}(f(x)) = F(k)$, then $\mathcal{F}(f'(x)) = ikF(k)$.

$$\int_{-\infty}^{\infty} f'(x) e^{-ikx} dx = f(x) e^{-ikx} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-ik) e^{-ikx} dx)$$
$$= ik F(k).$$

$$\mathcal{F}(f''(x)) = -k^2 F(k).$$

Convolution f, g

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy$$

$$\int_{-\infty}^{\infty} (f*g)(x) e^{-ikx} dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) g(y) dy \, e^{-ikx} dx$$

$$z = x - y \quad dz = dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z) e^{-ik(y+z)} g(y) \, dy \, dz$$

$$= \left(\int_{-\infty}^{\infty} f(z) e^{-ikz} dz \right) \left(\int_{-\infty}^{\infty} g(y) e^{-iky} dy \right)$$

$$= F(k) \cdot G(k) = \mathcal{F}(f(x)) \cdot \mathcal{F}(g(x))$$

Example 1 $\quad u_{xx} = \delta(x), \quad x \in \mathbb{R}.$

$$\hat{u}_{xx} = \hat{\delta} = -k^2 \hat{u} = 1 \Rightarrow \hat{u} = -\frac{1}{k^2} = \left(-\frac{i}{k}\right) \cdot \left(-\frac{i}{k}\right)$$

① We know $H(x) - H(-x) \xrightarrow[\mathcal{F}]{} \frac{2}{i} \frac{1}{k}$

Then $h(x) = \dfrac{H(x) - H(-x)}{2} = \dfrac{1}{ik} = -\dfrac{i}{k}$

So $u(x) = (h*h)(x) = \displaystyle\int_{-\infty}^{\infty} \frac{H(x-y) - H(y-x)}{2} \cdot \frac{H(y) - H(-y)}{2} dy$

② Another way

If $\hat{f} = F$, then $\widehat{xf} = i\frac{dF}{dk}$

If $\hat{h} = \frac{1}{ik}$ then $\widehat{xh} = \frac{-1}{ik^2}\cdot i = -\frac{1}{k^2}$

So $u = F^{-1}\left(-\frac{1}{k^2}\right) = xh = x\cdot\frac{H(x)-H(-x)}{2} = \frac{|x|}{2}$

Example 2 $\begin{cases} S_t = S_{xx} & x\in\mathbb{R},\ t>0. \\ S(x,0) = \delta(x) \end{cases}$

$\hat{S}(k,t) = \int_{-\infty}^{\infty} S(x,t)e^{-ikx}dx \Rightarrow \frac{d\hat{S}}{dt} = -k^2\hat{S}$, $\hat{S}(k,0) = 1$

$\hat{S}(k,t) = e^{-k^2t} =$

Formula (P345) (ii) $e^{-k^2/2} \xrightarrow[F^{-1}]{} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

(P346) (vi) $F\left(\frac{k}{a}\right) \xrightarrow[F^{-1}]{} |a|f(ax) \Rightarrow e^{-(k/a)^2/2} \rightarrow |a|\frac{1}{\sqrt{2\pi}}e^{-a^2x^2/2}$

Let $a^2 = \frac{1}{2t}$ then $e^{-k^2t} = e^{-(k^2/a^2)/2}$

$= e^{-(k^2/a^2)/2} \xrightarrow{F^{-1}} \frac{1}{\sqrt{2t}}\frac{1}{\sqrt{2\pi}}e^{-a^2x^2/2}$

$= \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$

So $S(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$

Example 3
$$\begin{cases} S_t = S_{xx}, & x \in \mathbb{R} \\ S(x,0) = f(x) \end{cases}$$

$$\frac{d\hat{S}}{dt} = -k^2 \hat{S}, \quad \hat{S}(k,0) = \hat{f} \quad \Rightarrow \quad \hat{S}(k,t) = \hat{f} e^{-k^2 t}$$

$$\Rightarrow S(k,t) = f * \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} = \int_{-\infty}^{\infty} f(x-y) \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}} dy$$

Example 4
$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in \mathbb{R}, y > 0, \\ u(x,0) = \delta(x), & x \in \mathbb{R}, y = 0 \end{cases}$$

$$U(k,y) = \int_{-\infty}^{\infty} e^{-ikx} u(x,y)\, dx$$

$$-k^2 U + U_{yy} = 0, \quad y > 0, \quad U(k,0) = 1.$$

Solution: $e^{\pm|k|y}$ we choose $e^{-|k|y}$ ($e^{|k|y} \to \infty$ for $y \to \infty$)

So $U(k,y) = e^{-|k|y}$

$$u(x,y) = F^{-1}(e^{-|k|y}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-y|k|} dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} e^{ikx} e^{ky} dk + \frac{1}{2\pi} \int_{0}^{\infty} e^{ikx} e^{-ky} dk$$

$$= \frac{1}{2\pi} \frac{1}{ix+y} e^{(ix+y)k} \Big|_{-\infty}^{0} + \frac{1}{2\pi} \cdot \frac{1}{ix-y} e^{(ix-y)k} \Big|_{0}^{\infty}$$

$$= \frac{1}{2\pi} \left(\frac{1}{ix+y} - \frac{1}{ix-y} \right) = \frac{1}{2\pi} \frac{ix-y-ix-y}{(ix+y)(ix-y)} = \frac{1}{2\pi} \frac{-2y}{-(x^2+y^2)} = \frac{y}{\pi(x^2+y^2)}$$

Green's function for half plane !

Example 5
$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in \mathbb{R}, y > 0 \\ u(x,0) = h(x), & x \in \mathbb{R}, y = 0. \end{cases}$$

$$U(k,y) = \int_{-\infty}^{\infty} e^{-ikx} u(x,y)\, dx$$

$$-k^2 U + U_{yy} = 0, \quad U(k,0) = \hat{h} \qquad U = e^{\pm|k|y}$$

$$U(k,y) = \hat{h} e^{-|k|y}$$

From Example 4 $\quad F^{-1}(e^{-|k|y}) = \dfrac{y}{\pi(x^2+y^2)}$

So $\quad F^{-1}(\hat{h} \cdot e^{-|k|y}) = h * \dfrac{y}{\pi(x^2+y^2)} = \displaystyle\int_{-\infty}^{\infty} \frac{h(x-z)\, y}{\pi(z^2+y^2)}\, dz$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{h(z)\, y}{(x-z)^2 + y^2}\, dz$$

HW
$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in \mathbb{R}, y > 0 \\ \dfrac{\partial u}{\partial y} = h(x), & x \in \mathbb{R}, y = 0. \end{cases}$$

$$-k^2 U + U_{yy} = 0 \qquad \frac{\partial U}{\partial y}(k,0) = \hat{h} \qquad U = c e^{-|k|y}$$

$$\frac{\partial U}{\partial y} = -c|k| e^{-|k|y}$$

$$\Rightarrow \frac{\partial U}{\partial y}(k,0) = -c|k| = \hat{h} \Rightarrow c = -\frac{\hat{h}}{|k|}$$

So $\quad U = -\dfrac{\hat{h}}{|k|} e^{-|k|y}$

Let $\quad v(x,y) = \dfrac{\partial u}{\partial y}(x,y) \qquad \begin{cases} v_{xx} + v_{yy} = 0 \\ v(x,0) = h(x) \end{cases}$ So apply Example 5! to get $\dfrac{\partial u}{\partial y}$