For distribution f defined on D_3 (3D)

partial derivative f_x is $(f_x, \phi) = -(f, \phi_x)$

f_{xx} is $(f_{xx}, \phi) = (f, \phi_{xx})$

Then Δf is $(\Delta f, \phi) = (f, \Delta\phi)$

So Δf is also a distribution.

Solve $\Delta u(\vec{x}) = \delta(\vec{x})$ in $\mathbb{R}^n$ $(n = 1, 2, 3)$

~~Green's formula:~~ $\cancel{\{\Delta u = f(x),\ x \in D}$

~~Recall~~ Let $\phi \in D_n$ be a test function,

Newton's potential formula (see Page 95 of notes)

$$\Rightarrow \underset{(\delta,\, \phi)}{\underset{\parallel}{\phi(0)}} = \int_{\mathbb{R}^n} \Gamma(x) \cdot \Delta\phi(x)\, dx = \underset{(\Delta\Gamma,\, \phi)}{\underset{\parallel}{\int_{\mathbb{R}^n} \Delta\Gamma(x) \cdot \phi(x)\, dx}}$$

So as distribution, $\Delta\Gamma = \delta$

So the fundamental solution $\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log|x| & n = 2 \\ -\frac{1}{4\pi} \cdot \frac{1}{|x|} & n = 3 \end{cases}$

is the solution of

$$\Delta\Gamma(x) = \delta(x), \quad x \in \mathbb{R}^n$$

What about $\mathbb{R}^1$?

We solve $u''(x) = \delta(x)$ directly

$$\Rightarrow u'(x) = H(x) + C_1 \qquad (H(x) = \text{Heaviside function})$$

$$\Rightarrow u(x) = p(x) + C_1 x + C_2 \qquad p(x) = \begin{cases} 0 & x<0 \\ x & x>1 \end{cases}$$

$$p(x) - \frac{1}{2}x = \begin{cases} -\frac{1}{2}x & x<0 \\ \frac{1}{2}x & x>0 \end{cases} = \frac{1}{2}|x|$$

So $\Gamma(x) = \frac{1}{2}|x|$ in $\mathbb{R}^1$. (see Homework 9)

Similarly the Green's function $G(x,x_0)$ satisfies

$$\begin{cases} \Delta G(x,x_0) = \delta(x-x_0), & x \in D \\ G(x,x_0) = 0, & x \in \partial D \end{cases}$$

Diffusion Equation

$$\begin{cases} u_t = k u_{xx}, & x \in \mathbb{R}, t>0 \\ u(x,0) = \delta(x) \end{cases}$$

Solution $S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$

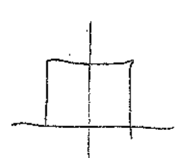
Wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R}, t>0 \\ u(x,0) = 0, \ u_t(x,0) = \delta(x) \end{cases}$$

d'Alembert formula $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \delta(y)\,dy$

$$= \frac{1}{2c}\left[H(x+ct) - H(x-ct)\right] = \begin{cases} \frac{1}{2c} & |x|<ct \\ 0 & |x|>ct \end{cases}$$

$\Rightarrow$ expending square wave.



Fourier Transform

For $f(x)$ defined on $(-l, l)$, Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}, \quad C_n = \frac{1}{2l}\int_{-l}^{l} f(y) e^{-in\pi y/l}\, dy$$

This can be thought as a transform

input : $\underbrace{\text{function } f(x)}_{\text{domain continuous}}$ output : $\underbrace{\{C_n\} \text{ sequence}}_{\text{domain discrete}}$ (~~frequency~~)

Coefficient C_n corresponds frequency $k = \frac{n\pi}{l}$

$$f(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[\int_{-l}^{l} f(y) e^{-iky}\, dy \right] e^{ikx} \cdot \frac{\pi}{l}$$

Let $l \to \infty$ so $\frac{\pi}{l} \to 0$, in the limit

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) e^{-iky} dy \right) e^{ikx}\, dk$$

<u>Fourier transform</u> of $f(x)$, $x \in (-\infty, \infty)$

$$F(k) = \int_{-\infty}^{\infty} f(y) e^{-iky}\, dy = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

Fourier transform is an operator from $\underbrace{\text{functions of } x}_{\text{state space}}$ to $\underbrace{\text{function of } k}_{\text{frequency space}}$.

<u>Inverse Fourier transform</u> $\quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk$

Example 1 $f(x)=1$

$$F(k)=\int_{-\infty}^{\infty} e^{-ikx}\,dx = \lim_{M\to\infty}\int_{-M}^{M} e^{-ikx}\,dx = \frac{1}{-ik}e^{-ikx}\Big|_{x=-M}^{x=M}$$

$$=\frac{1}{-ik}\left(e^{-ikM}-e^{ikM}\right)=\frac{1}{-ik}\left(\cos kM - i\sin kM - \cos kM - i\sin kM\right)$$

$$=\frac{-2i\sin kM}{-ik} = \frac{2\sin kM}{k} = ? \quad (M\to\infty)$$

Let $F_M(k)=\dfrac{2\sin kM}{k}$. Then F_M is a distribution, We take limit of F_M in the space of distributions. Let $\phi\in D_1$ (test function)

$$\int_{-\infty}^{\infty} F_M(k)\phi(k)\,dk = \int_{-\infty}^{\infty}\frac{2\sin kM}{k}\phi(k)\,dk \underset{\downarrow}{=} \int_{-a}^{a}\frac{2\sin kM}{k}\phi(k)\,dk$$

$(\phi(k)=0$ for $|k|>a)$

Let $kM=y$. $$=\int_{-aM}^{aM}\frac{2\sin(y)}{y}\phi\left(\frac{y}{M}\right)dy \xrightarrow{M\to 0} \phi(0)\int_{-\infty}^{\infty}\frac{2\sin y}{y}\,dy$$

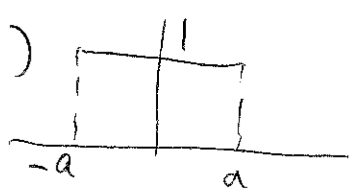
$$\boxed{\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=\pi}$$

$$=2\pi\phi(0) = 2\pi\delta(k)$$

So $\mathcal{F}(1)=2\pi\delta(k)$!!!

Example 2

Square wave (square pulse)



$$\mathcal{F}(H(a-|x|)) = \frac{2\sin ak}{k}$$ (same calculation as above)

Example 3 $f(x)=e^{-x^2/2}$

$y=x+ik$

$$\int_{-\infty}^{\infty}e^{-x^2/2}e^{-ikx}\,dx = \int_{-\infty}^{\infty}e^{-(x+ik)^2/2}dx\cdot e^{i^2k^2/2} = e^{-k^2/2}\int_{-\infty}^{\infty}e^{-y^2/2}dy$$

$$=e^{-k^2/2}\sqrt{2}\int_{-\infty}^{\infty}e^{-y^2}dy = \sqrt{2\pi}\,e^{-k^2/2}$$

$$\mathcal{F}\left(e^{-x^2/2}\right)=\sqrt{2\pi}\,e^{-k^2/2}$$