Math 442 Lecture 21

Mapping $f: A \to B$, A and B are set

(function) $\forall x \in A$, $\exists\, y = f(x) \in B$ (a rule of assigning ~~a number~~ an element in B)

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functional $B = \mathbb{R}$ (assigning a number $\forall x \in A$)

$f: \mathbb{R} \to \mathbb{R}$. $f(x) = x^2$

$f: \mathbb{R}^n \to \mathbb{R}$ $f(\vec{x}) = |\vec{x}| = \sqrt{\sum_{i=1}^{n} x_i^2}$, $f(\vec{x}) = \vec{a} \cdot \vec{x}$

$f: C[0,1] \to \mathbb{R}$ $f(u) = \int_0^1 u(x)\,dx$ (area)

$f(u) = \int_0^1 \sqrt{1 + u'(x)^2}\,dx$ (arc length)

$f(u) = u(\frac{1}{2})$

Operator B = another space (probably not $\mathbb{R}$, could be A)

$f: \mathbb{R}^n \to \mathbb{R}^n$ $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$ $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

linear operator

$f(u)(x)$

$f: C[0,1] \to C[0,1]$ $f(u(x)) = \int_0^x u(t)\,dt$

$f: C^1[0,1] \to C[0,1]$ $f(u)(x) = u'(x)$

Delta function is a functional, X is a space of functions.

$\delta: X \to \mathbb{R}$ $\delta(u(x)) = u(0)$.

δ function: input (a function) output (the value at 0 of that function)

X is a space of functions defined on $\mathbb{R}$.

In comparison with δ-function, we consider another kind of functional

$\phi_f : X \to \mathbb{R}$ $\quad \phi_f(u(x)) = \int_{-\infty}^{\infty} f(x)\,u(x)\,dx$ where

$f : \mathbb{R} \to \mathbb{R}$ is a fixed function. So any $f(x)$ induces a functional

$\phi_f : X \to \mathbb{R}$.

Now what function induces $\delta : X \to \mathbb{R}$?

ϕ_f is an average function with a weight $f(x)$. (non-local)

δ is a functional only needs value at 0 (and ignore any other values)

(highly local)

$$\delta_f : \mathbb{R} \to \overline{\mathbb{R}} \qquad \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$$

$$\delta_f(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

generalized function

$$\int_{-\infty}^{\infty} \delta_f(x)\,dx = 1$$

(Since value $= \infty$)

$$\delta(u(x)) = \int_{-\infty}^{\infty} \delta_f(x)\,u(x)\,dx = u(0)$$

Now we officially define X.

① A <u>test function</u> $\phi(x)$ is a C^∞ function defined on $\mathbb{R}$, and $\phi(x) = 0$ outside of a finite interval.

D = the set of all test functions.

② A <u>distribution</u> f is a functional $f: D \to \mathbb{R}$. [P112

which is linear and continuous. (f, ϕ) is the value of f at a test function ϕ

<u>linear</u>: $a, b \in \mathbb{R}$, $\phi, \psi \in D$, $(f, a\phi + b\psi) = a(f, \phi) + b(f, \psi)$

<u>Continuity</u>: If $\{\phi_n\} \subseteq D$, ϕ_n converges to ϕ <u>uniformly</u>, $\phi \in D$,

then $\lim_{n\to\infty} (f, \phi_n) = (f, \phi)$ ↳ means $\underbrace{\phi_n^{(k)}}_{\text{derivatives}} \to \phi^{(k)}$

<u>Example 1</u> Define $(\delta, \phi) = \phi(0)$, Then δ is a distribution

<u>linear</u> Yes $(\delta, a\phi) = a\phi(0) = a(\delta, \phi)$, $(\delta, \phi + \psi) = \phi(0) + \psi(0)$

<u>Continuous</u> if $\phi_n \to \phi$, then $\phi_n(0) \to \phi(0)$ (of course if $\max|\phi_n - \phi| \to 0$)

<u>Example 2</u> If $f(x)$ is an integrable function defined on $\mathbb{R}$

Define $(f, \phi) = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$ Then f "is" a distribution.

<u>linear</u> ✓ <u>continuous</u> if $\phi_n \to \phi$, then $\int_{-\infty}^{\infty} f(x)\phi_n(x)\,dx \to \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$

So any function is a distribution (in some sense)

But not ~~all~~ every distribution defines a function (δ) <u>generalized functions</u>

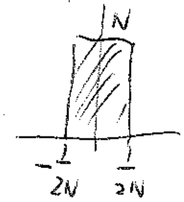
$$(\delta, \phi) = \int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx = \phi(0)$$

<u>Convergence of distributions</u>

$\{f_N\}$ a sequence of distributions, we say f_N <u>converges weakly</u> to f

if $\forall \phi \in D$, $\lim_{N\to\infty} (f_N, \phi) = (f, \phi)$

Example 3 Define $f_N(x) = \begin{cases} N & |x| < \frac{1}{2N} \\ 0 & |x| \geq \frac{1}{2N} \end{cases}$



Take $\phi \in D$ $\quad \int_{-\infty}^{\infty} f_N(x)\phi(x)\,dx = \int_{-\frac{1}{2N}}^{\frac{1}{2N}} N\phi(x)\,dx \xrightarrow[N\to\infty]{} N\cdot\frac{1}{N}\phi(0) = \phi(0)$

Since ϕ is continuous at $x=0$.

So $f_N \to \delta \quad (N\to\infty)$

Example 4 Define $S_N(x) = \frac{\sqrt{N}}{\sqrt{4\pi k}} e^{-\frac{Nx^2}{4k}} \quad \left(S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}},\ t = \frac{1}{N} \right)$

Take $\phi \in D$

$$\int_{-\infty}^{\infty} S_N(x)\phi(x)\,dx = \frac{\sqrt{N}}{\sqrt{4\pi k}} \int_{-\infty}^{\infty} e^{-\frac{Nx^2}{4k}} \phi(x)\,dx \qquad y = \sqrt{\frac{N}{4k}}\,x,\quad dy = \sqrt{\frac{N}{4k}}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} \phi\left(\sqrt{\frac{4k}{N}}\,y\right) dy \xrightarrow[N\to\infty]{} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2}\phi(0)\,dy = \phi(0)$$

So $S_N \to \delta \quad (N\to\infty)$

(The limit of functions may not be a function, but a distribution / generalized function)

When $t\to 0$ $\quad S(x,t) \xrightarrow[t\to 0]{} \delta(x)$

$$\int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy \xrightarrow[t\to 0]{} \int_{-\infty}^{\infty} \delta(x-y)\,\phi(y)\,dy = \phi(x)$$

$\downarrow$ $\qquad$ So it satisfies the initial condition!

Solves

$$\begin{cases} u_t = k u_{xx}, & x\in\mathbb{R} \\ u(x,0) = \phi(x) \end{cases}$$

$$S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$$ (Fundamental solution of diffusion equation)

Solves

$$\begin{cases} u_t = k u_{xx}, & x \in \mathbb{R}, t > 0 \\ u(x,0) = \delta(x) \end{cases}$$

Derivative of a distribution (defined through integral by parts!)

$\forall \phi \in \mathcal{D}$. $f \in C^1(\mathbb{R})$ $\int_{-\infty}^{\infty} f'(x)\phi(x)\,dx = -\int_{-\infty}^{\infty} f(x)\phi'(x)\,dx$

We define a distribution f' by

$$(f', \phi) = -(f, \phi')$$

and we call f' to be the (weak) derivative of $f \in \mathcal{D}$

This way any distribation has any order of derivative

$$(f'', \phi) = -(f', \phi') = \underline{(f, \phi'')}$$

Heaviside function $H(x) = \begin{cases} 0 & x<0 \\ 1 & x>0. \end{cases}$

$$(H', \phi) = -(H, \phi') = -\int_{-\infty}^{\infty} H(x)\phi'(x)\,dx = -\int_0^{\infty} \phi'(x)\,dx = -\phi(x)\Big|_0^{\infty} = \phi(0)$$

We know $(\delta, \phi) = \phi(0)$ So $H'(x) = \delta(x)$!

Plus function $P(x) = \begin{cases} 0 & x<0 \\ x & x>0 \end{cases} = x^+$

$$P'(x) = H(x), \quad H'(x) = \delta(x) \quad P''(x) = \delta(x)$$

Fourier series of $\delta(x)$

Homework : $|x| = \frac{\pi}{2} - \sum_{n \text{ odd}} \frac{4}{n^2\pi} \cos(nx)$ $\quad x \in [-\pi, \pi]$

diff $\quad 2H(x) - 1 = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin(nx)$

diff $\quad 2\delta(x) = \sum_{n \text{ odd}} \frac{4}{\pi} \cos(nx)$

So $\quad \delta(x) = \sum_{n \text{ odd}} \frac{2}{\pi} \cos(nx) = \frac{2}{\pi} \sum_{n \text{ odd}} \cos(nx)$

$$\Rightarrow \sum_{n \text{ odd}} \int_{-\pi}^{\pi} \phi(x) \cos(nx)\,dx = \frac{\pi}{2} \phi(0).$$

So far in $\mathbb{R}$, But in $\mathbb{R}^2$, or $\mathbb{R}^3$, test functions distributions can also be defined.

Let D_n be the set of test functions in $\mathbb{R}^n$ $(n=1,2,3)$

$(C^\infty$ in $\mathbb{R}^n$, vanishing outside a ball in $\mathbb{R}^n)$

Let X_n be the set of distributions on D_n $(n=1,2,3)$

(linear, continuous functional $D_n \to \mathbb{R}$)

Then Delta function $\delta(u) = u(0) \quad \forall u \in D_n$

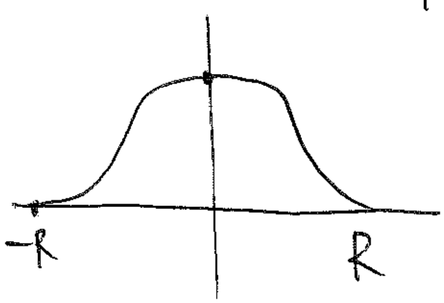
Is there a test function?

$f \in C^\infty(\mathbb{R})$, and $f(x) \equiv 0$ for $|x| \geq R$.

$f(x) = 0$ of cource. but what else?

<u>Lemma</u> $g(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$ is C^∞ and $g^{(n)}(0) = 0$

Define $f_R(x) = \begin{cases} e^{\frac{1}{x^2 - R^2}} & -R < x < R \\ 0 & \text{otherwise} \end{cases}$



$g_R(x) = f_R(x)\, g(x)$

for any $g(x) \in C^\infty(\mathbb{R})$

$g(x) = e^x$, $\sin x$, $\cos x$, polynomial, ...