Math 442 Lecture 20

Fourier's method in higher dimension

$$u_{tt} = c^2 \Delta u \quad \text{or} \quad u_t = k\Delta u \qquad \vec{x} = (x,y,z) \in D \subset \mathbb{R}^3 \ (\mathbb{R}^2)$$

$$u(\vec{x},t) = T(t)\,v(\vec{x})$$

$$-\lambda = \frac{T'(t)}{kT(t)} = \frac{\Delta v(\vec{x})}{v(\vec{x})} \Rightarrow \begin{cases} -\Delta v(\vec{x}) = \lambda v(\vec{x}), & \text{in } D \\ \text{B.C.} & \text{on } \partial D \end{cases}$$

$\Rightarrow$ eigenvalue λ_n, eigenfunction $v_n(\vec{x})$

$$\Rightarrow u(\vec{x},t) = \sum_n A_n e^{-\lambda_n k t} v_n(\vec{x})$$

Thm 1 Eigenvalue $\lambda_n \geq 0$, $\lambda_n \in \mathbb{R}$,

Eigenvalues Problem in $D \subset \mathbb{R}^2$

① $D = (0,a)\times(0,b)$ rectangle.

$$\begin{cases} -(u_{xx}+u_{yy}) = \lambda u, & (x,y)\in(0,a)\times(0,b) \\ u=0 & (x,y)\in\partial D \end{cases}$$

$$u(x,y) = U(x)V(y)$$

$$-\frac{U''(x)V(y)+U(x)V''(y)}{U(x)V(y)} = \lambda \Rightarrow -\frac{U''(x)}{U(x)} - \frac{V''(y)}{V(y)} = \lambda$$

Let $\lambda = \mu + \theta$ $\quad \begin{cases} U''(x)+\mu U(x) = 0 \\ U(0) = U(a) = 0 \end{cases} \quad \begin{cases} V''(y)+\theta V(y) = 0 \\ V(0) = V(b) = 0 \end{cases}$

$$\mu_n = \frac{n^2\pi^2}{a^2}, \quad n=1,2,\cdots \quad U_n(x) = \sin\left(\frac{n\pi}{a}x\right)$$

$$\theta_m = \frac{m^2\pi^2}{b^2}, \quad m=1,2,\cdots \quad V_m(y) = \sin\left(\frac{m\pi}{b}y\right)$$

$$\lambda_{nm} = \mu_n + \theta_m = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right), \quad n, m \in \mathbb{N}$$

double index $\qquad u_{nm}(x,y) = \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)$

~~if $a=\pi$, $b=\pi/2$, $\lambda_{nm} = n^2 + 4m^2$~~

If $a=b=\pi$ (square), then $\lambda_{nm} = n^2+m^2$.

$$\lambda_{11} = 2 < \lambda_{12} = \lambda_{21} = 5 < \lambda_{22} = 8 < \lambda_{13} = \lambda_{31} = 10 < \cdots$$

$\lambda_{11} = 2$ simple eigenvalue — dimension of eigenspace $= 1$ — $\sin x \sin y$

$\lambda_{12} = \lambda_{21} = 5$ double eigenvalue — $= 2$.

$\sin(x)\sin(2y)$, $\sin(2x)\sin(y)$

triple eigenvalue? — Could be $1^2+7^2 = 5^2+5^2 = 50$

$n^2+m^2 = l^2+k^2$. — $15^2+20^2 = 25^2 = 7^2+24^2 = 625$

$n^3+m^3 = l^3+k^3$? — $1^3+12^3 = 9^3+10^3 = 1729$ Ramanujan number

$$\begin{cases} u_t = k(u_{xx}+u_{yy}), & 0<x<a,\ 0<y<b \\ u = 0 & (x,y)\in\partial D \\ u(x,y,0) = \phi(x,y) \end{cases}$$

$$u(x,y,t) = \sum_{n,m=1}^{\infty} A_{nm}\, e^{-k\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)t} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)$$

$$A_{nm} = \frac{\int_0^a\int_0^b \phi(x,y)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)dy\,dx}{\int_0^a\int_0^b \sin^2\left(\frac{n\pi}{a}x\right)\sin^2\left(\frac{m\pi}{b}y\right)dy\,dx}$$

(2) $D = \{x^2+y^2<a^2\}$ disk

$$\begin{cases} -(u_{xx}+u_{yy}) = \lambda u, & (x,y)\in D \\ u=0, & (x,y)\in\partial D \end{cases}$$

$$u(r,\theta) = R(r)v(\theta)$$

$$-\left(u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}\right) = \lambda u$$

$$\cancel{-\frac{R''(r)v(\theta)+\frac{1}{r}R'(r)v(\theta)+\frac{1}{r^2}R(r)v''(\theta)}{R(r)v(\theta)} = \frac{\lambda}{R}} \quad -\frac{R''}{R}-\frac{R'}{rR}-\frac{v''}{r^2v}=\lambda$$

$$\Rightarrow \frac{v''}{v} = -\gamma, \quad -\frac{R''}{R}-\frac{R'}{rR}+\frac{\gamma}{r^2}=\lambda$$

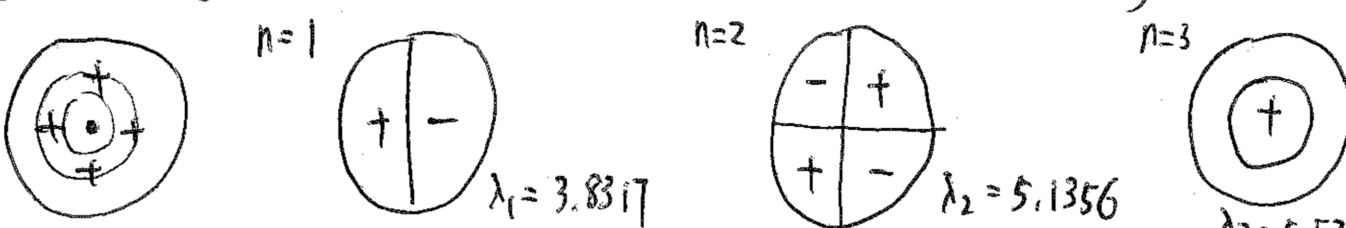
$$\begin{cases} v''+\gamma v=0 \\ v(\theta)=v(\theta+2\pi) \\ v'(\theta)=v'(\theta+2\pi) \end{cases} \Rightarrow \gamma_n = n^2 \qquad n=0,1,2,\cdots$$

$$v_n(\theta) = A_n\cos\theta + B_n\sin\theta$$

$$\begin{cases} R''+\frac{1}{r}R'+\left(\lambda-\frac{\gamma}{r^2}\right)R=0, & 0<r<a \\ R(0)<\infty,\quad R'(0)=0,\quad R(a)=0 \end{cases}$$

Solution: <u>Bessel functions</u> for $\gamma_n=n^2 \Rightarrow \lambda_{nm}$ eigenvalues

<u>Example</u> Eigenvalues of unit disk $D\subset\mathbb{R}^2$ $(a=1)$

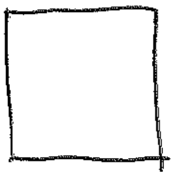


$\lambda_0=2.4048$ $\quad \lambda_1=3.8317$ $\quad \lambda_2=5.1356$ $\quad \lambda_3=5.5201$

λ_{01} $\quad \lambda_{11}$ $\quad \lambda_{21}$ $\quad \lambda_{02}$

Area of $D=\pi$

Compare it with a square with same area



$\sqrt{\pi}$ $\quad$ $\sqrt{\pi}$

$$\lambda_{mn} = \left(\frac{n^2}{\pi} + \frac{m^2}{\pi}\right)\pi^2 = \pi(n^2+m^2)$$

$$\lambda_0 = 2\pi = 6.28\cdots, \quad \lambda_1 = 5\pi = 15.71, \quad \lambda_2 = 8\pi = 25.13 \cdots$$

Compare with a rectange with sides 1 and π

1 $\quad$ π

$$\lambda_{mn} = \left(\frac{n^2}{\pi^2} + \frac{m^2}{1^2}\right)\pi^2 = n^2 + m^2\pi^2$$

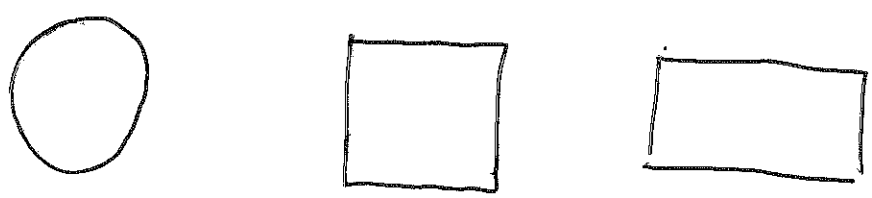
$$\lambda_0 = 1+\pi^2 = 10.87, \quad \lambda_1 = 4+\pi^2 = 13.87, \quad \lambda_3 = 9+\pi^2 = 18.87 \cdots\cdots$$

In wave equation
$$\begin{cases} u_{tt} = c^2(u_{xx}+u_{yy}), & (x,y)\in D \\ u=0 & (x,y)\in\partial D \end{cases}$$

$$u(x,y,t) = \sum_n \left(A_n\cos(c\sqrt{\lambda_n}t) + B_n\sin(c\sqrt{\lambda_n}t)\right)\phi_n(x,y)$$

So λ_n represents the frequencies of wave (sound, $\cdots$)

For example, vibration of a drumhead



λ_0	2.4048	6.28	10.87
λ_1	3.8317	15.71	13.87
λ_2	5.1356	25.13	18.87

Rayleigh Conjecture (1877)

Let D be a domain in $\mathbb{R}^n$, and let B_n be the unit ball in $\mathbb{R}^n$.

If $Vol(D) = Vol(B_n)$, then $\lambda_1(D) \geq \lambda_1(B_n)$

proved by Faber–Krahn (1923) $\quad$ So the ball minimizes the first eigenvalues

A related question : isoperimetric problem

Let D be a domain in $\mathbb{R}^2$, and let B_2 be the unit disk in $\mathbb{R}^2$.

If $\text{per}(D) = \text{per}(B_2)$, the $\text{Vol}(D) \geq \text{Vol}(B_2)$.

Famous question #2 about eigenvalues

Kac (1966) "Can one hear the shape of a drum ?"

$\{\lambda_n : \text{eigenvalues}\} \Rightarrow$ Shape of $D \subset \mathbb{R}^2$?

Or if D_1 and D_2 have same eigenvalues for all $n \geq 1$, then do D_1 and D_2 have exact same shape ? (Up to rotation, reflection, ...)

Gordon-Webb-Wolpert (1992) $\Rightarrow$ No !

Famous question #3 about eigenvalues

$$\begin{cases} -\Delta u = \lambda u, & \text{in } D \subset \mathbb{R}^2 \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \end{cases}$$

Neumann boundary problem

$\lambda_0 = 0$, $u_0 = 1$, $\lambda_1 > 0$, $u_1(x)$ eigenfunction

Rauch (1975) The maximum and minimum of $u_1(x)$ are on ∂D.

"Hotest spot is always on the boundary." (hot spot conjecture)

$$\begin{cases} u_t = d\Delta u \\ \frac{\partial u}{\partial n} = 0 \end{cases} \Rightarrow u(x,t) = \frac{1}{2}A_0 + \sum_n A_n e^{-d\lambda_n t} u_n(x)$$

$$\max_x u(x,t) \approx \max_x u_p(x)$$

Math 442 Homework 9: (due April 17 (Friday), 2015)

1. Page 184 Problem 5: Prove the Dirichlet's principle for the Neumann problem. It asserts that among all real-valued functions $w(x)$ on D the quantity

$$E[w] = \frac{1}{2}\int_D |\nabla w(x)|^2 dx - \int_{\partial D} h(x)w(x)dS,$$

is the smallest for $w = u$, where u is the solution for the Neumann problem

$$\begin{cases} -\Delta u = 0, & \text{in } D, \\ \dfrac{\partial u}{\partial n} = h(x), & \text{on } \partial D \end{cases}$$

It is required to assume that the average of the given function $h(x)$ is zero, *i.e.* $\int_D h(x)dx = 0$. (Hint: follow the method in Section 7.1 (Page 183), or my notes)

2. Page 196 Problem 1: Find the one-dimensional Green's function $G(x, x_0)$ for the interval $(0, l)$. The three properties defining it can be stated as follows:

   (i) It solves $G_{xx}(x, x_0) = 0$ for $x \neq x_0$.

   (ii) $G(0, x_0) = G(l, x_0) = 0$.

   (iii) $G(x, x_0)$ is continuous at x_0 and $G(x, x_0) + \frac{1}{2}|x - x_0|$ is harmonic at x_0.

   (Hint: Let $G(x, x_0) = -\frac{1}{2}|x - x_0| + ax + b$)

3. Page 196 Problem 3: From Page 192 (3), show directly that the boundary condition is satisfied:

$$\lim_{z_0 \to 0} u(x_0, y_0, z_0) = h(x_0, y_0).$$

Assume that $h(x, y)$ is continuous and bounded. (Hint: Change variables $s^2 = [(x - x_0)^2 + (y - y_0)^2]/z_0^2$ and use the fact $\int_0^\infty s(s^2 + 1)^{-3/2} ds = 1$)

4. Page 196 Problem 6:

   (a) Find the Green's function for the half-plane $D = \{(x, y) : y > 0\}$. (Hint: this is done in class)

   (b) Use it to solve the Dirichlet problem in the half plane with boundary values $h(x)$, that is

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u(x, y) = h(x), & \text{on } \partial D = \{y = 0\} \end{cases}$$

   (c) Calculate the solution with $u(x, 0) = 1$, and get a formula without integral.

5. Page 197 Problem 17:

   (a) Find the Grenn's function in the first quadrant $Q = \{(x, y) : x > 0, y > 0\}$. (Hint: use the method of reflection)

   (b) Use your answer in part (a) to solve the Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{in } Q, \\ u(0, y) = g(y), & y > 0, \\ u(x, 0) = h(x), & x > 0. \end{cases}$$