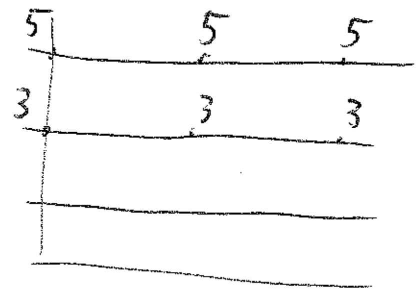
Math 442 Lecture 2

1.2 1st order linear PDE

① Simpliest case : $u_x = 0$:

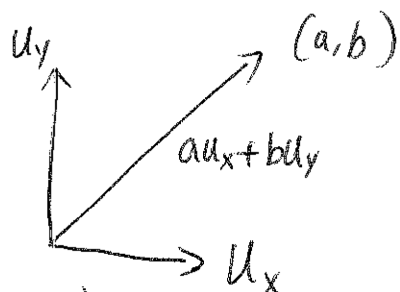


The value of $u(x,y)$ does not change when x chages.

$$\int u_x\,dx = \int 0\,dx \Rightarrow u(x,y) = f(y)$$

where $f(y)$ is an arbitrary function. (may be not differentiable)

② $au_x + bu_y = 0$



$\boxed{\text{review of multi-variable calc I}}$:

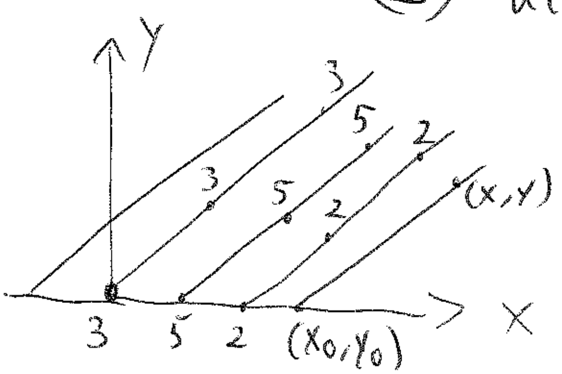
u_x : derivative in direction $(1,0)$

u_y : derivative in direction $(0,1)$

au_x+bu_y : derivative in direction (a,b) → (strictly speaking, (a,b) should satisfy $\sqrt{a^2+b^2}=1$)

$au_x+bu_y = 0 \iff$ the directional derivative along (a,b) $= 0$

$\iff u(x,y)$ is a constant along the direction (a,b)



line $y = \frac{b}{a}x + C$

$y - y_0 = \frac{b}{a}(x - x_0)$ → characteristic curve

$ay - ay_0 = bx - bx_0 \Rightarrow ay - bx = ay_0 - by_0$

So $u(x,y) = u(x_0, y_0)$ if $ay - bx = ay_0 - bx_0$

$\Rightarrow u(x,y) = f(ay - bx)$ where f is an arbitrary differentiable function

Example 1 initial value problem

$$\begin{cases} 3u_t - 5u_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = \sin x, & x \in \mathbb{R}. \end{cases}$$

direction : $(a,b) = (3,-5)$

$$u(t,x) = f(ax - bt) = f(3x + 5t)$$

when $t = 0$, $u(0,x) = f(3x) = \sin x$

Let $3x = w$ then $x = \frac{w}{3} \Rightarrow f(w) = \sin \frac{w}{3}$

Therefore $u(t,x) = \sin\left(\frac{3x+5t}{3}\right)$

Recipe 1 : $au_x + bu_y = 0$

characteristic curve $ay - bx = c$

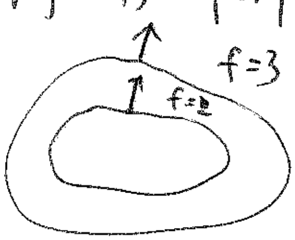
solution $u(x,y) = f(ay - bx)$

review of m-var calc 2 For a function $f(x,y)$

gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is the direction f increasing fastest

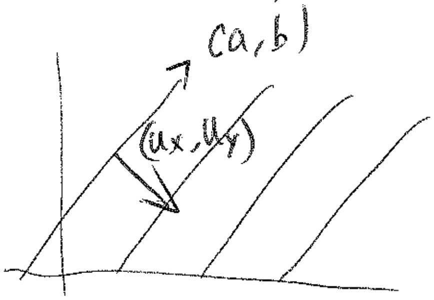
level curve $f(x,y) = C$

∇f is perpendicular to the level curve



~~For $f(x,y) = ax + by$ $\nabla f = (a,b)$~~

$a u_x + b u_y = 0 \Rightarrow u(x,y) = f(ay - bx)$



level curve

$(a,b) \cdot (u_x, u_y) = 0$

So (u_x, u_y) is in direction $(b, -a)$

For $u(x,y) = f(ay-bx)$

$u_x = f'(ay-bx)\cdot(-b) \quad u_y = f'(ay-bx)\cdot a$

So $a u_x + b u_y = 0$

So characteristic curve is the level curve for $a u_x + b u_y = 0$.

③ general linear 1st order PDE.

$a(x,y)\, u_x + b(x,y)\, u_y = 0$ or $u_x + c(x,y)\, u_y = 0$

$\Rightarrow$ directional derivative along $(1, c(x,y))$ is 0

$\Rightarrow$ Solution $u(x,y)$ along an orbit of $\frac{dy}{dx} = c(x,y)$ is a constant

If $y(x)$ is a solution of $\frac{dy}{dx} = c(x,y)$, then a function like

$U(x, y(x))$ satisfies

$\frac{dU}{dx} = U_x + U_y \cdot \frac{dy}{dx} = U_x + U_y\, c(x,y) = 0$.

Recipe 2 : $u_x + c(x,y)\, u_y = 0$

Step 1 : Solve characteristic curve $\frac{dy}{dx} = c(x,y)$

Solution $g(x,y) = C$

Step 2 : Solution : $u(x,y) = f(g(x,y))$ for an arbitrarily differentiable function f

Example 2 $\quad xu_x + yu_y = 0$ (page 10 #5)

$$u_x + \frac{y}{x} u_y = 0$$

characteristic ODE : $\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{dx}$

$$\Rightarrow \ln y = \ln x + C \Rightarrow y = Cx \Rightarrow \frac{y}{x} = C \quad \left(g(x,y) = \frac{y}{x}\right)$$

Solution: $u(x,y) = f\left(\frac{y}{x}\right)$

Example 3

$$\begin{cases} xu_x + yu_y = 0, & x > 1, \ y \in \mathbb{R}, \\ u(1,y) = \cos y, & y \in \mathbb{R}. \end{cases}$$

$u(x,y) = f\left(\frac{y}{x}\right) \qquad u(1,y) = f(y) = \cos y$

So $u(x,y) = \cos\left(\frac{y}{x}\right)$

Example 1 (again)

$$\begin{cases} 3u_t - 5u_x = 0, & t > 0, \ x \in \mathbb{R} \\ u(0,x) = \sin x. \end{cases}$$

$u_t - \frac{5}{3} u_x = 0 \qquad \frac{dx}{dt} = -\frac{5}{3} \Rightarrow x = -\frac{5}{3}t + C \Rightarrow x + \frac{5}{3}t = C$

$u(t,x) = f\left(x + \frac{5}{3}t\right)$

$u(0,x) = f(x) = \sin x \Rightarrow u(t,x) = \sin\left(x + \frac{5}{3}t\right)$

Example 4

$$\begin{cases} u_t + t^2 u_x = 0, & t > 0, \ x \in \mathbb{R}, \\ u(0,x) = e^{-x^2}, & x \in \mathbb{R}. \end{cases}$$

characteristic ODE : $\frac{dx}{dt} = t^2. \quad \Rightarrow dx = t^2 dt \Rightarrow x = \frac{1}{3}t^3 + C$

$$\Rightarrow x - \frac{1}{3}t^3 = C$$

$u(t,x) = f\left(x - \frac{1}{3}t^3\right) \quad \overset{t=0}{} \quad u(0,x) = f(x) = e^{-x^2}$

$$\Rightarrow u(t,x) = e^{-\left(x - \frac{1}{3}t^3\right)^2}$$

Example 5 (page 10 #7)

1) Solve $\begin{cases} y u_x + x u_y = 0, \\ u(0,y) = e^{-y^2}, \end{cases}$

2) In which region of then xy-plane, is the solution uniquely determined?

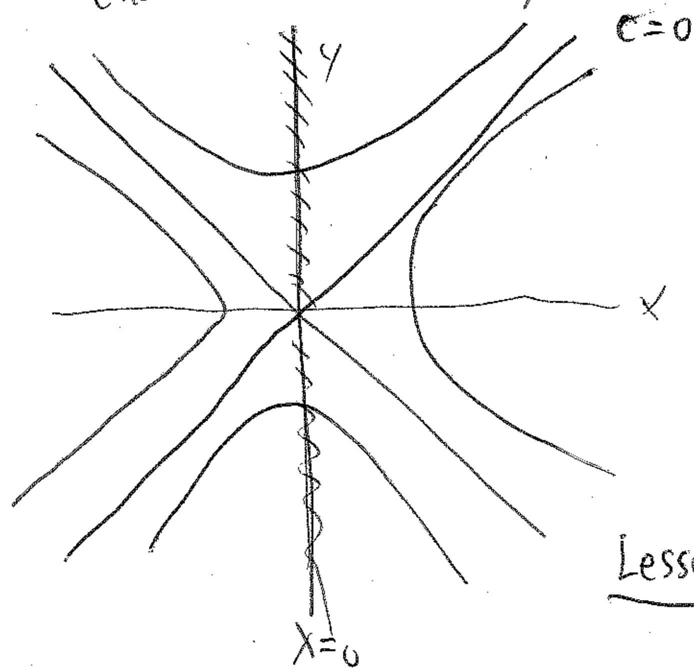
$$u_x + \frac{x}{y} u_y = 0 \quad \frac{dy}{dx} = \frac{x}{y} \Rightarrow y\,dy = x\,dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$\Rightarrow y^2 - x^2 = C \quad \Rightarrow u(x,y) = f(y^2 - x^2)$$

$$x = 0 \quad u(0,y) = e^{-y^2} = f(y^2) \Rightarrow f(y) = e^{-y}$$

$$\text{So } u(x,y) = e^{-(y^2 - x^2)} = e^{x^2 - y^2}$$

characteristic curve $y^2 - x^2 = C$

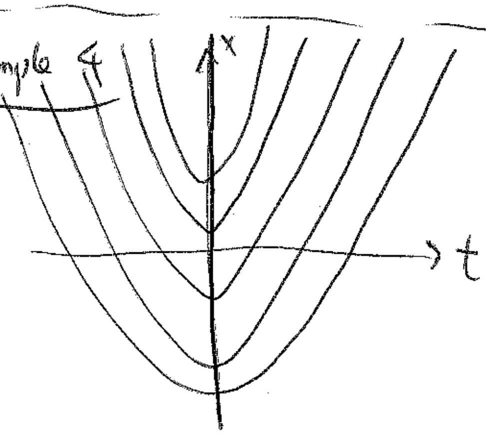


in the region $|y| \geq |x|$,
the solution is determined by
initial condition $u(0,y) = e^{-y^2}$

but in the region $|y| < |x|$,
Solution can be arbitrary.

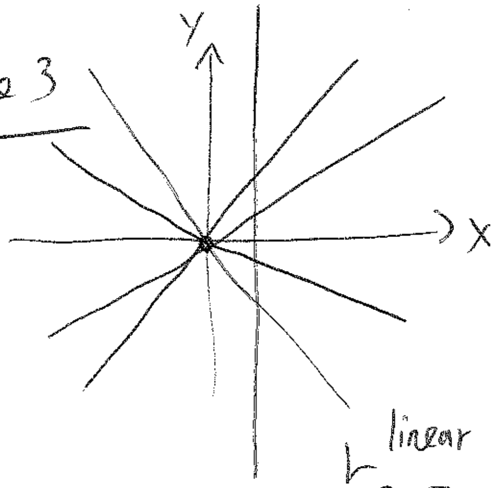
Lesson: geometry of characteristic curves determines where the solution is uniquely determined.

Example 4



all $\mathbb{R}^2$ is uniquely determined

Example 3



determines all $x>0$, $y \in \mathbb{R}$

Application of 1st order linear PDE

① Pollution model (transport equation)

a river flows at a constant velocity c along x-direction

$u(x,t)$ = density of a pollutant

amount of pollutant in $[0, y]$ at time $t = \int_0^y u(x,t)\,dx$

$[cs, cs+y]$ at time $t+s = \int_{cs}^{cs+y} u(x,t+s)\,dx$

balance law $\rightarrow \int_0^y u(x,t)\,dx = \int_{cs}^{cs+y} u(x,t+s)\,dx$

$\frac{d}{dy}$ $\quad u(y,t) = u(y+cs, t+s)$

$\frac{d}{ds}$ $\quad 0 = c\,u_x(y+cs, t+s) + u_t(y+cs, t+s)$

$s=0$ $\quad 0 = c\,u_x(y,t) + u_t(y,t)$

$$\Rightarrow \quad u_t(x,t) + c\,u_x(x,t) = 0$$

So linear 1st order PDE with constant velocity

shows the transport of substance from one location to another