Math 442 Lecture 17

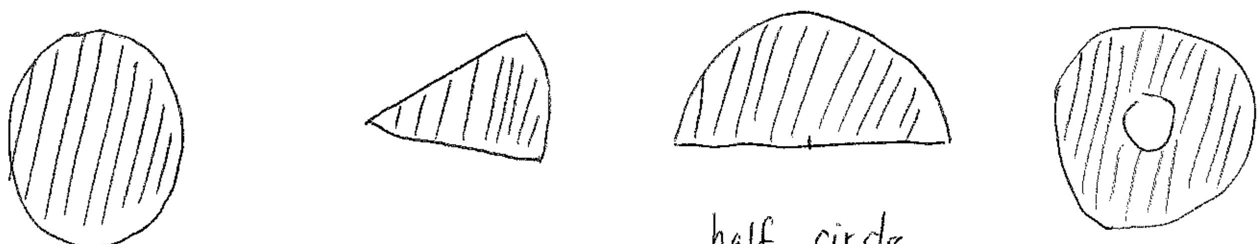
Similar to Poisson's formula (Circular region)

~~Circle~~ Disk : $0 \le r \le a$, $0 \le \theta \le 2\pi$ (with $0 = 2\pi$).

Wedge : $0 \le r \le a$, $0 \le \theta \le \beta$ ($\beta < 2\pi$)

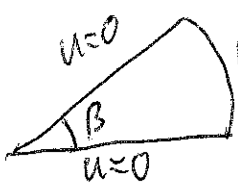
$\beta = \pi \Rightarrow$ semicircle (homework) $\quad \beta = \frac{\pi}{2} \Rightarrow$ quarter circle.

annulus $\quad 0 < a \le r \le b$, $0 \le \theta \le 2\pi$ (with $0 = 2\pi$)



Disk $\qquad$ wedge (pizza slice) $\qquad$ half circle $\qquad$ annulus.

Wedge



$u_r = h(\theta) \qquad u(r,\theta) = R(r)V(\theta)$

$$\begin{cases} V'' + \lambda V = 0, & 0 < \theta < \beta \\ V(0) = V(\beta) = 0 \end{cases} \Rightarrow \lambda_n = \left(\frac{n\pi}{\beta}\right)^2, \quad V_n(\theta) = \sin\left(\frac{n\pi\theta}{\beta}\right)$$

R satisfies $r^2R'' + rR' - \lambda R = 0$ for $R(r) = r^\alpha \Rightarrow \alpha = \pm\frac{n\pi}{\beta} \Rightarrow R_n(r) = r^{\frac{n\pi}{\beta}}$

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi\theta}{\beta}\right) \qquad u_r(r,\theta) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{\beta} r^{\frac{n\pi}{\beta}-1} \sin\left(\frac{n\pi\theta}{\beta}\right)$$

$$u_r(a,\theta) = h(\theta) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{\beta} a^{\frac{n\pi}{\beta}-1} \sin\left(\frac{n\pi\theta}{\beta}\right)$$

$$\Rightarrow A_n \frac{n\pi}{\beta} a^{\frac{n\pi}{\beta}-1} \int_0^\beta \left(\sin\frac{n\pi\theta}{\beta}\right)^2 d\theta = \int_0^\beta h(\theta)\sin\frac{n\pi\theta}{\beta}\, d\theta.$$

half disk : $\beta = \pi$, Quarter disk: $\beta = \frac{\pi}{2}$

annulus $\{a<r<b\}$: $R_n(r) = C_n r^n + D_n r^{-n}$, ($r^{-n}$ only has singularity at $r=0$)

exterior of a disk : $\{r>a\}$ $R_n(r) = D_n r^{-n}$ (r^n is unbounded)

All can be solved similarly as disk or wedge.

Chap 8 Green's Identities and Green's Functions

Notations

$f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z)$	$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ (vector field)
$\nabla f: \mathbb{R}^3 \to \mathbb{R}^3$ (f_x, f_y, f_z) (gradient)	$= (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$
	$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$. (divergence)

$\nabla \cdot \nabla f = \Delta f = \operatorname{div}(\nabla f) = f_{xx} + f_{yy} + f_{zz}$ (Laplacian)

$\nabla f \underset{\text{inner product}}{\cdot} \nabla f = |\nabla f|^2 = f_x^2 + f_y^2 + f_z^2$

integral in $D \subseteq \mathbb{R}^3$

$$\iiint_D \cdots dx\,dy\,dz = \int_D \cdots d\vec{x} \qquad \vec{x} = (x, y, z)$$

integral on S (a surface in $\mathbb{R}^3$), like ∂D

$$\iint_S \cdots dS = \int_S dS \qquad \text{(Surface integral)}$$

Divergence Theorem $\iiint_D \operatorname{div} \vec{F}\, d\vec{x} = \iint_{\partial D} \vec{F}\cdot\vec{n}\, dS$

$\vec{F}$: vector field , $\vec{n}$ = unit outer normal direction on ∂D.

Divergence Theorem is one of generalizations of Fundamental Theorem of Calculus in 1D

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Integral by parts (from product rule)

$$\int_a^b [u'(x)v(x) + u(x)v'(x)]\,dx = u(b)v(b) - u(a)v(a)$$

$$\Rightarrow \int_a^b u v'\,dx = uv\Big|_a^b - \int_a^b u' v\,dx$$

3D version of integral by parts $\Rightarrow$? product rule in vector

For $u, v: \mathbb{R}^3 \to \mathbb{R}$

$$\cancel{\nabla(u\cdot v) = ((uv)_x, (uv)_y, (uv)_z)}$$

We take $\vec{F} = v\nabla u = (vu_x, vu_y, vu_z)$

$$\begin{aligned}
\operatorname{div}(v\nabla u) &= (vu_x)_x + (vu_y)_y + (vu_z)_z \\
&= v_x u_x + vu_{xx} + v_y u_y + vu_{yy} + v_z u_z + vu_{zz} \\
&= \nabla v\cdot\nabla u + v\cdot\Delta u
\end{aligned}$$

From Divergence Theorem

$$\iiint_D \operatorname{div}(v\nabla u)\,d\vec{x} = \iiint_D (\nabla v\cdot\nabla u + v\cdot\Delta u)\,d\vec{x}$$

$$\|$$

$$\iint_{\partial D} v\nabla u\cdot\vec{n}\,dS = \iint_{\partial D} v\cdot\frac{\partial u}{\partial n}\,dS$$

Green's First Identity

$$\iiint_D v\Delta u \, d\vec{x} = \iint_{\partial D} v\frac{\partial u}{\partial n}\, ds - \iiint_D \nabla v\cdot\nabla u \, d\vec{x} \qquad (1)$$

Now switch u, and v

$$\iiint_D u\Delta v \, d\vec{x} = \iint_{\partial D} u\frac{\partial v}{\partial n}\, ds - \iiint_D \nabla u\cdot\nabla v \, d\vec{x} \qquad (2)$$

(1) − (2) $\Rightarrow$

$$\iiint_D (v\Delta u - u\Delta v)\, d\vec{x} = \iint_{\partial D} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds$$

Green's Second Identity

1D version

$$\int_a^b v u_{xx}\, dx = v u_x \Big|_a^b - \int_a^b u_x v_x \, dx$$

$$\int_a^b (v u_{xx} - u v_{xx})\, dx = (v u_x - u v_x)\Big|_a^b$$

Special version $(u = u,\ v = 1)$

$$\iiint_D \Delta u \, d\vec{x} = \iint_{\partial D} \frac{\partial u}{\partial n}\, ds$$

Homework 2 : $\begin{cases} \Delta u = f(x) & \text{in } D \\ \dfrac{\partial u}{\partial n} = g(x) & \text{on } \partial D \end{cases} \Rightarrow \iiint_D f(x)\, d\vec{x} = \iint_{\partial D} g(x)\, dS$

Another proof of mean-value property (2D and 3D)

Let u satisfy $\Delta u = 0$ in D. Take a ball centered at x_0 : B

$$0 = \iiint_B \Delta u \, d\vec{x} = \iint_{\partial B} \frac{\partial u}{\partial n}\, ds = \int_0^{2\pi}\int_0^{\pi} u_r(a,\theta,\phi)\, a^2 \sin\theta \, d\theta\, d\phi$$

This holds for any $a > 0$.

Define $F(r) = \int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)\sin\theta\, d\theta\, d\phi$ (average value on sphere)

Then $F'(r) = \int_0^{2\pi}\int_0^{\pi} u_r(r,\theta,\phi)\sin\theta\, d\theta\, d\phi = 0$

$\Rightarrow F(r) =$ constant for $r>0$. Let $r \to 0$

$\Rightarrow F(0) = 4\pi u(0,0,0) = 4\pi u(0) = F(r)$

$$\Rightarrow \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)\, r^2 \sin\theta\, d\theta\, d\phi = u(0)$$

$$\Rightarrow \frac{1}{\text{area of } \partial B}\iint_{\partial B} u(\vec{x})\, ds = u(0)$$ (also harmonic in $\mathbb{R}^3$)

Uniqueness (energy method) $\begin{cases} \Delta u = f(x) & \text{in } D \\ u = g(x), & \text{on } \partial D \end{cases}$

Let $w = u_1 - u_2$. $\Rightarrow \begin{cases} \Delta w = 0 & \text{in } D \\ w = 0 & \text{on } \partial D \end{cases}$

Apply Green's First Identity to $w \cdot \Delta w$

$$0 = \int_D w\Delta w\, d\vec{x} = \int_{\partial D} w\frac{\partial w}{\partial n}\, ds - \int_D \nabla w\cdot\nabla w\, d\vec{x} = -\int_D |\nabla w|^2\, d\vec{x}$$

So from vanishing theorem $\nabla w = 0$ $\forall x \in D \Rightarrow w(\vec{x}) = C$, $\forall x \in D$

But $w(\vec{x}) = 0$ on ∂D Then $w(\vec{x}) = 0$, $\forall x \in D$.

Dirichlet Principle Define an energy function $E(w) = \frac{1}{2}\int_D |\nabla w|^2\, d\vec{x}$.

where $w \in X = \{ w \in C^2(\overline{D}) : w(x) = g(x) \text{ on } \partial D \}$

Then $\min_{w\in X} E(w)$ is achieved by the harmonic function

$$\begin{cases} \Delta w = 0 & x \in D, \\ w(x) = g(x) & x \in \partial D. \end{cases}$$

~~Suppose $W \in X$ is the function achieving $\min_{w\in X} E(w)$~~

~~Take~~ Let w be that harmonic function (which is unique as we have just show) and let v be any other function with same boundary value. We show that $E(v) \geq E(w)$. Let $v = w + u$

$$E(w+u) = \frac{1}{2}\int_D |\nabla(w+u)|^2 d\vec{x} = \frac{1}{2}\int_D \nabla(w+u)\cdot\nabla(w+u)\, d\vec{x}$$

$$= \frac{1}{2}\int_D \left(|\nabla w|^2 + 2\nabla w\cdot\nabla u + |\nabla u|^2\right) d\vec{x}$$

Now $\int_D \nabla w\cdot\nabla u = \int_{\partial D} u\cdot\frac{\partial w}{\partial n}\, ds - \int_D u\cdot\Delta w\, d\vec{x} = 0 - 0 = 0$

Since $u\big|_{\partial D} = v\big|_{\partial D} - w\big|_{\partial D} = g(x) - g(x) = 0$

$\Delta w = 0$

So $E(w+u) = \frac{1}{2}\int_D |\nabla w|^2 d\vec{x} + \frac{1}{2}\int_D |\nabla u|^2 d\vec{x} = E(w) + E(u)$

$> E(w)$.

So harmonic function minimizes the energy $\frac{1}{2}\int_D |\nabla u|^2 d\vec{x}$

<u>History</u>: The above proof is correct if

1. Such a harmonic function w exists.
2. $\min_{w\in X} E(w)$ can be achieved.

① $\Rightarrow$ Yes if ∂D is "smooth"

② $\Rightarrow$ Yes if X is a larger space than $C^2(D)$

Dirichlet (1870s)
Riemann, Hilbert