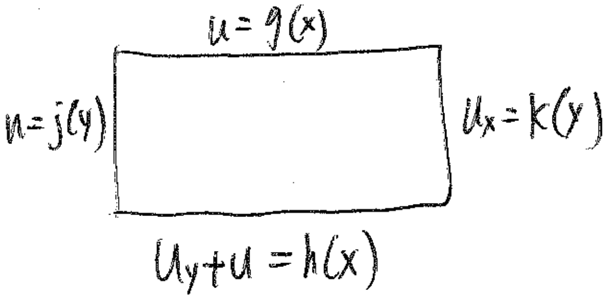
Math 442 Lecture 16

Laplace equation in a rectangle

$$\begin{cases} u_{xx} + u_{yy} = 0, & (x,y) \in (0,a)\times(0,b) \\ \text{B.C.} \end{cases}$$



$\text{B.C} = (h, g, j, k)$

Superposition Principle : Let u_1, u_2, u_3, u_4 be the solution with B.C. $(h,0,0,0)$, $(0,g,0,0)$, $(0,0,j,0)$, $(0,0,0,k)$

Then $u = u_1 + u_2 + u_3 + u_4$ (Textbook $\rightarrow u_2$, #5 $\rightarrow u_3$)

We solve u_1:

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u = 0, & x = 0,\ 0 \le y \le b \\ u_x = 0, & x = a,\ 0 \le y \le b \\ u_y + u = h(x), & 0 \le x \le a,\ y = 0 \\ u = 0, & 0 \le x \le a,\ y = b \end{cases}$$

$$X(0)\cdot Y(y) = 0$$
$$X'(0)\cdot Y(y) = 0$$
$$X(x)\cdot (Y'(0) + Y(0)) = h(x)$$
$$X(x)\cdot Y(b) = 0$$

$u(x,y) = X(x)\cdot Y(y)$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0, \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < a, \\ X(0) = 0, & X'(0) = 0 \end{cases} \qquad \begin{cases} Y'' - \lambda Y = 0, & 0 < y < b, \\ Y(b) = 0 \end{cases}$$

$\Downarrow$

Solution : $\lambda_n = \dfrac{(n+\frac{1}{2})^2\pi^2}{a^2}$, $X_n(x) = \sin \dfrac{(n+\frac{1}{2})\pi x}{a}$

$$\beta_n = \left(n+\tfrac{1}{2}\right)\frac{\pi}{a}$$

$$\begin{cases} Y_n'' - \beta_n^2 Y_n = 0 \\ Y_n(b) = 0 \end{cases} \Rightarrow Y_n(y) = A_n \cosh(\beta_n y) + B_n \sinh(\beta_n y)$$

$$A_n \cosh(\beta_n b) + B_n \sinh(\beta_n b) = 0$$

Let $A_n = \sinh(\beta_n b)$, $B_n = -\cosh(\beta_n b)$.

So $$u(x,y) = \sum_{n=0}^{\infty} C_n \sin(\beta_n x)\left(A_n \cosh(\beta_n y) + B_n \sinh(\beta_n y)\right)$$

with $\beta_n = \left(n+\frac{1}{2}\right)\frac{\pi}{a}$

$$U_y(x,y) = \sum_{n=0}^{\infty} C_n \sin(\beta_n x)\beta_n\left(A_n \sinh(\beta_n y) + B_n \cosh(\beta_n y)\right)$$

$$(U_y + u)(x,0) = \sum_{n=0}^{\infty} C_n \sin(\beta_n x)\left(A_n + \beta_n B_n\right) = h(x)$$

$\{\sin(\beta_n x)\}$ is an orthonormal basis.

$$C_n(A_n + \beta_n B_n)\int_0^a \sin^2(\beta_n x)\,dx = \int_0^a h(x)\sin(\beta_n x)\,dx$$

$$C_n = \frac{\int_0^a \sin(\beta_n x)\, h(x)\,dx}{(A_n + \beta_n B_n)\int_0^a \sin^2(\beta_n x)\,dx}.$$

Finally solved !!

So $\begin{cases} \Delta u = 0 \\ B.C \end{cases}$ is solvable if D is a rectangle

$D = \text{disk}$ in $\mathbb{R}^2$

$$\begin{cases} u_{xx} + u_{yy} = 0, & x^2+y^2<a^2 \\ u = h(\theta), & x^2+y^2=a^2. \end{cases}$$

① convert to polar coordinate.

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

② separate the variables $u(r,\theta) = R(r)v(\theta)$

$$0 = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = R''(r)v(\theta) + \frac{1}{r}R'(r)v(\theta) + \frac{1}{r^2}R(r)v''(\theta)$$

dividing by $R(r)v(\theta)$

$$\Rightarrow \quad 0 = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{v''}{V} \Rightarrow 0 = \underbrace{r^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right)}_{\text{depending on } r} + \underbrace{\frac{v''}{V}}_{\text{depending on } \theta}$$

Let $\quad r^2\frac{R''}{R} + r\frac{R'}{R} = \lambda = -\frac{V''}{V}$

$$\begin{cases} v'' + \lambda v = 0, & 0<\theta<2\pi \\ V(0) = V(2\pi),\ V'(0) = V'(2\pi) \end{cases}$$

periodic boundary condition
= Solution on a circle?

Solved before

$\lambda_n = n^2, \quad n = 0, 1, 2, \cdots \quad V_n(\theta) = A_n\cos n\theta + B_n\sin n\theta.$

Now we solve $\quad r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = \lambda$

$$\Rightarrow \cancel{r R + r} \; r^2R'' + rR' - \lambda R = 0, \qquad 0<r<a$$

Euler type ODE

$\underline{\lambda = n^2}$ $\quad r^2 R'' + rR' - n^2 R = 0$

guess a solution $R(r) = r^\alpha$ $\quad R'(r) = \alpha r^{\alpha-1}$ $\quad R''(r) = \alpha(\alpha-1) r^{\alpha-2}$

$$r^2 \cdot \alpha(\alpha-1) r^{\alpha-2} + r \cdot \alpha r^{\alpha-1} - n^2 r^\alpha = 0.$$

$$\Rightarrow \alpha(\alpha-1) + \alpha - n^2 = 0 \Rightarrow \alpha^2 - n^2 = 0 \Rightarrow \alpha = \pm n.$$

So $R_n(r) = C_n r^n + D_n r^{-n}$ $\quad r^{-n}$ causes a singularity at $r=0$!

So $R_n(r) = C_n r^n$

$\underline{\lambda = 0}$ $\quad r^2 R'' + rR' = 0$ $\quad R'' + \frac{1}{r} R' = 0$ (solved last time)

$R_0(r) = C_0 + D_0 \log r$ $\quad$ So $R_0(r) = C_0$

<u>series solution</u> $\quad u(r,\theta) = \sum R_n(r) V_n(\theta)$

$$= \frac{1}{2} A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos n\theta + B_n \sin n\theta \right)$$

<u>matching BC</u> $\quad h(\theta) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} a^n \left(A_n \cos n\theta + B_n \sin n\theta \right)$

$$\Rightarrow A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(s) \cos(ns)\, ds \quad , \quad B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(s) \sin(ns)\, ds$$

$$\left(A_0 = \frac{1}{\pi} \int_0^{2\pi} h(s)\, ds \right)$$

<u>Formula</u> $\quad u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} h(\phi)\, d\phi + \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n} \int_0^{2\pi} h(\phi) \left(\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta) \right) d\phi$

$$= \frac{1}{2\pi} \int_0^{2\pi} h(\phi) \left\{ 1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n\theta - n\phi) \right\} d\phi \rightarrow \text{Dirichlet kernel!}$$

(from Sec. 5.5)

$$1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos(n\theta-n\phi)$$

$$=1+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left(e^{in(\theta-\phi)}+e^{-in(\theta-\phi)}\right)$$

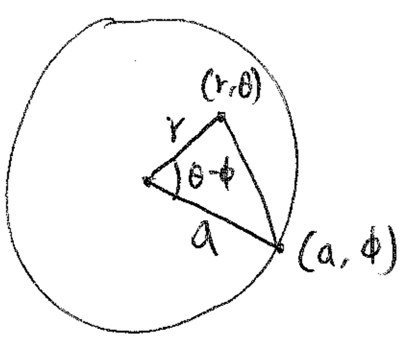
$$=1+\frac{\frac{r}{a}e^{i(\theta-\phi)}}{1-\frac{r}{a}e^{i(\theta-\phi)}}+\frac{\frac{r}{a}e^{-i(\theta-\phi)}}{1-\frac{r}{a}e^{-i(\theta-\phi)}}=1+\frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}}+\frac{re^{-i(\theta-\phi)}}{a-re^{-i(\theta-\phi)}}$$

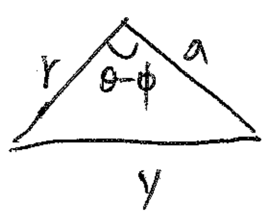
Let $\theta-\phi=\eta$

$$=1+\frac{r(\cos\eta+i\sin\eta)}{a-r(\cos\eta+i\sin\eta)}+\frac{r(\cos\eta-i\sin\eta)}{a-r(\cos\eta-i\sin\eta)}=\cdots$$

$$=\frac{a^2-r^2}{a^2-2ar\cos\eta+r^2}$$ So we have a formula without summation!

$$u(r,\theta)=\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\phi)}{a^2-2ar\cos(\theta-\phi)+r^2}\,d\phi$$





$$y^2=a^2-2ar\cos(\theta-\phi)+r^2$$

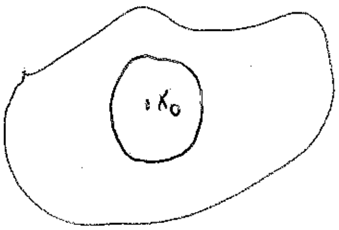
Law of Cosine

Let $\vec{x}=(r,\theta)$, $\vec{y}=(a,\phi)$

$$u(\vec{x})=\frac{a^2-|\vec{x}|^2}{2\pi a}\int_{|\vec{y}|=a}\frac{u(\vec{y})}{|\vec{x}-\vec{y}|^2}\,ds'$$ Poisson's formula

① Mean value property of harmonic function

Let u be a harmonic function in a region $D \subseteq \mathbb{R}^2$. Then for any disk $C \subseteq D$ with center $\vec{X}_0$, $u(\vec{X}_0)$ = average of u on the circle ∂C (also the average of disc)



Applying Poisson's formula to $u(\vec{x})$ with $\vec{x} = 0$

$$u(0) = \frac{a^2}{2\pi a} \int_{|\vec{y}|=a} \frac{u(\vec{y})}{|\vec{y}|^2}\, ds'$$

$$= \frac{1}{2\pi a} \int_{|\vec{y}|=a} u(\vec{y})\, ds'$$

$2\pi a$ = perimeter of circle with radius a

You can always assume $\vec{X}_0 = 0$ by shifting the coordinate.

This is what harmonic means.

② Maximum Principle If u is harmonic in $D \subseteq \mathbb{R}^2$, and u attains the maximum value at $\vec{x}_0 \in \bar{D}$. Then $\vec{x}_0 \in \partial D$ and $\vec{x}_0 \notin \partial D$, unless $u(\vec{x}) \equiv$ constant for $\vec{x} \in \bar{D}$.

proof We have proved $u(\vec{x}) \leq \max_{\vec{y} \in \partial D} u(\vec{y})$. Now if $\vec{x}_0 \in D$, we can draw a circle C around $\vec{x}_0$ which is also inside D, then from mean value property

$$\max_{\vec{x} \in \bar{D}} u(\vec{x}) = u(\vec{x}_0) = \frac{1}{2\pi a} \int_{|\vec{y}|=a} u(\vec{y})\, ds' \leq \frac{1}{2\pi a} \cdot 2\pi a \max_{\vec{x} \in \bar{D}} u(\vec{x}) = \max_{\vec{x} \in \bar{D}} u\vec{x}$$

Then $u(\vec{x}) = \max_{\vec{x} \in \bar{D}} u(\vec{x})$ for all $\vec{x} \in C$. This way we can prove

$u(\vec{x}) = \max$ for all $\vec{x} \in D$.