Math 442 Lecture 15

higher dimension
wave equation $u_{tt} = c^2 \Delta u$ diffusion equation $u_t = k \Delta u$

$$\begin{aligned} \Delta u &= u_{xx} \quad (1\text{-d}) \\ &= u_{xx} + u_{yy} \quad (2\text{-d}) \\ &= u_{xx} + u_{yy} + u_{zz} \quad (3d) \end{aligned}$$

Laplace operator Δu

Steady state $u_{tt} = 0$, $u_t = 0$ $u(x,t) = u(x)$ (independent of time)

Laplace equation $\Delta u = 0$

Solution of Laplace equation = harmonic functions $\begin{cases} u(x) \\ u(x,y) \\ u(x,y,z) \end{cases}$

Chap 6 Harmonic functions

1-d (so easy) $u(x) = A + Bx$ (solving $u''(x) = 0$)

2-d $u_{xx} + u_{yy} = 0$ possible solutions: $u(x,y) = Ax + By + C$, $x^2 - y^2$, $2xy$, $\sin x \cdot e^y$, $\sinh x \cdot \sin y$ .... so many.

You can view it as a wave equation $u_{xx} = (-1) u_{yy}$ or

$$u_{xx} = i^2 u_{yy}$$

D'Lembert $\Rightarrow$ $u(x,y) = f(x+iy) + g(x-iy)$

and f, g are any C^2 functions

Real part and imaginary part are both harmonic functions.

① $f(x) = x^n$ $f(x+iy) = (x+iy)^n$.

So $\mathrm{Re}((x+iy)^n)$ and $\mathrm{Im}((x+iy)^n)$ are harmonic

$(x+iy)^2 = x^2+2ixy-y^2 \Rightarrow x^2-y^2, \ 2xy$

$(x+iy)^3 = x^3+3x^2yi-3xy^2-y^3i \Rightarrow x^3-3xy^2, \ 3x^2y-y^3$

These are <u>harmonic polinomials</u>

② $f(x) = e^{kx}$ $\quad f(x+iy) = e^{kx}(\cos ky + i\sin ky)$

$e^{kx}\cos ky, \ e^{kx}\sin ky, \ \cdots$ (More in Math 405 Complex analysis)

<u>Conclusion</u>: There are so many harmonic functions! (in 2-d, also 3-d)

<u>polar coordinate</u> (2D) $\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$ $\quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$u_{xx}+u_{yy} = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta} = 0$$

$u(r,\theta)$ $\quad r =$ radius, $\theta =$ angle

<u>radially symmetric solution</u> $u(r)$ (independent of θ)

$$u_{rr}+\frac{1}{r}u_r = 0 \Rightarrow ru_{rr}+u_r = 0 \Rightarrow (ru_r)_r = 0 \Rightarrow ru_r = C_1$$

$$\Rightarrow u_r = \frac{C_1}{r} \Rightarrow u(r) = C_1 \ln r + C_2$$

<u>Spherical coordinate</u> (3D)

$$u_{xx}+u_{yy}+u_{zz} = u_{rr}+\frac{2}{r}u_r+\frac{1}{r^2}\left[u_{\theta\theta}+\cot\theta\cdot u_\theta+\frac{1}{\sin^2\theta}u_{\phi\phi}\right]$$

$$\begin{cases} x = r\sin\theta\cos\phi & 0 \le \theta < 2\pi \\ y = r\sin\theta\sin\phi & r > 0 \\ z = r\cos\theta & 0 \le \phi \le \pi \end{cases}$$

In general, Boundary value problem (no initial value)

Laplace equation
$$\begin{cases} \Delta u = 0 & x \in D \rightarrow \text{in the region } D \\ u = h,\ \dfrac{\partial u}{\partial n} = h,\ \dfrac{\partial u}{\partial n} + au = h, & x \in \partial D \rightarrow \text{on the boundary } \partial D \end{cases}$$

Dirichlet Newmann Robin

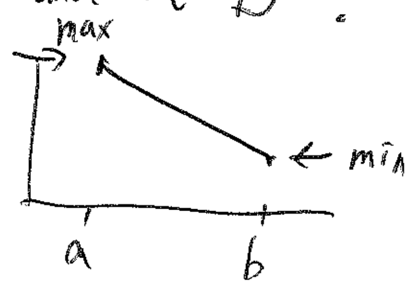
Poisson Equation $\Delta u = f$

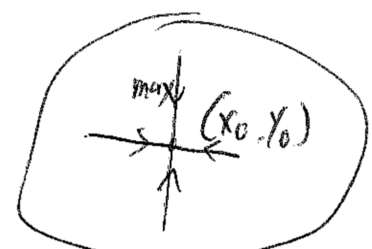
$$\Delta u(x,y,z) = f(x,y,z)$$

Maximum Principle

Let D be a connected bounded open set in $\mathbb{R}^2$ or $\mathbb{R}^3$, and let $u(x,y)$ or $u(x,y,z)$ be a harmonic function in D that is continuous on $\overline{D} = D \cup \partial D$. Then the maximum of the minimum of u are attained on ∂D and not D.

1D: $u(x) = Ax + B$





2D: If $u(x,y)$ achieves its maximum at (x_0, y_0) which is inside D, then $u_x(x_0,y_0) = u_y(x_0,y_0) = 0$.

$u_{xx}(x_0,y_0) \leq 0$ and $u_{yy}(x_0,y_0) \leq 0$.

But $u_{xx} + u_{yy} = 0$. Then $u_{xx}(x_0,y_0) = u_{yy}(x_0,y_0) = 0$.

radial solution $u_{rr} + \frac{2}{r} u_r = 0 \Rightarrow r^2 u_{rr} + 2r u_r = 0$

$$\Rightarrow (r^2 u_r)_r = 0 \Rightarrow r^2 u_r = C_1 \Rightarrow u_r = \frac{C_1}{r^2} \Rightarrow u(r) = -\frac{C_1}{r} + C_2$$

Example (1) Find a radial solution of $u_{xx} + u_{yy} = 0$ in $D = \{(x,y) : 1 < x^2+y^2 < 2\}$ with $u(x,y) = 1$ on $r = 1$, $u(x,y) = 0$ on $r = 2$.

$u(r) = C_1 \ln r + C_2$ $\quad u(1) = C_2 = 1 \quad u(2) = C_1 \ln 2 + C_2 = 0$

$$\Rightarrow C_2 = 1, \quad C_1 = -\frac{1}{\ln 2} \Rightarrow u(r) = -\frac{\ln r}{\ln 2} + 1$$

$$u(x,y) = -\frac{\ln\sqrt{x^2+y^2}}{\ln 2} + 1 = -\frac{\ln(x^2+y^2)}{2\ln 2} + 1$$

(2) Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell $a < r < b$ with u vanishing on both inner and outer boundaries

$u(r) = -\frac{C_1}{r} + C_2$? Solution = particular solution + harmonic function

$$u_{xx} + u_{yy} + u_{zz} = 1 \Rightarrow u(x,y,z) = \frac{1}{6}(x^2+y^2+z^2) = \frac{1}{6}r^2$$

So $u(r) = \frac{1}{6}r^2 - \frac{C_1}{r} + C_2$ $\quad u(a) = \frac{1}{6}a^2 - \frac{C_1}{a} + C_2 = 0$

$$u(b) = \frac{1}{6}b^2 - \frac{C_1}{b} + C_2 = 0$$

$$\Rightarrow \frac{1}{6}(a^2-b^2) - C_1\left(\frac{1}{a} - \frac{1}{b}\right) = 0 \Rightarrow \frac{1}{6}(a-b)(a+b) + \frac{C_1}{ab}(a-b) = 0$$

$$\Rightarrow C_1 = -\frac{(a+b)ab}{6}, \quad C_2 = -\frac{1}{6}a^2 - \frac{(a+b)b}{6} = -\frac{a^2+ab+b^2}{6}$$

Then $$u(r) = \frac{1}{6}r^2 + \frac{(a+b)ab}{6r} - \frac{a^2+ab+b^2}{6}$$

$\vec{x} = (x, y)$

proof Define $u_\varepsilon(\vec{x}) = u(\vec{x}) + \varepsilon |\vec{x}|^2$ $\quad |\vec{x}|^2 = x^2 + y^2$

Then $\Delta u_\varepsilon = \Delta u + 4\varepsilon = 4\varepsilon > 0$ in D

If u_ε has an interior maximum $(x_\varepsilon, y_\varepsilon)$, then $\Delta u_\varepsilon(x_\varepsilon, y_\varepsilon) \le 0$

That is a contradiction! So u_ε achieves its maximum at $\vec{x}_\varepsilon \in \partial D$. So $\forall \vec{x} \in D$

$$u(\vec{x}) \le u_\varepsilon(\vec{x}) \le u_\varepsilon(\vec{x}_\varepsilon) = u(\vec{x}_\varepsilon) + \varepsilon |\vec{x}_\varepsilon| \le \max_{\vec{x} \in \partial D} u(\vec{x}) + \varepsilon l^2$$

where l = max distance from ∂D to the origin.

This is true for all $\varepsilon > 0$, so let $\varepsilon \to 0$, we get

$$u(\vec{x}) \le \max_{x \in \partial D} u(\vec{x})$$

$\blacksquare$

Uniqueness of solution of Laplace equation or Poisson Equation for Dirichlet BVP.

$$\begin{cases} \Delta u = f(x), & \text{in } D \\ u = h, & \text{on } \partial D \end{cases}$$ has at most one solution.

proof Suppose u_1, u_2 are two solutions. Then $w = u_1 - u_2$ satisfies

$$\begin{cases} \Delta w = 0 & \text{in } D \\ w = 0 & \text{on } \partial D \end{cases}$$

$$\underbrace{\min_{\vec{y} \in \partial D} w(\vec{y})}_{=0} \le w(\vec{x}) \le \underbrace{\max_{\vec{y} \in \partial D} w(\vec{y})}_{=0} \quad \text{for } \vec{x} \in D$$ So $w(\vec{x}) = 0$, $\forall \vec{x} \in D$.

$$\Rightarrow u_1(\vec{x}) = u_2(\vec{x})$$