Math 442 Lecture 13

Type of BC:

Dirichlet	$X(a) = X(b) = 0$
Neumann	$X'(a) = X'(b) = 0$
Periodic	$X(-l) = X(l)$, $X'(-l) = X'(l)$
Robin	$\cancel{X'(a) - a_0 X(b) = 0,\ X'(a) + a_l X(b) = 0}$
	$X'(a) - a_0 X(a) = 0$, $X'(b) + a_l X(b) = 0$

$$\begin{cases} u_t = k u_{xx}, & a<x<b \\ BC \\ u(x,0) = \phi(x), \end{cases} \Rightarrow \text{eigenvalue problem} \begin{cases} -X'' = \lambda X, & a<x<b \\ BC. \end{cases}$$

$\Rightarrow$ eigenvalue λ_n $(n = 1, 2, 3 \cdots)$

eigenfunction $X_n(x)$

Solution $$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-k\lambda_n t} X_n(x)$$

$$C_n = \frac{\int_a^b \phi(x) X_n(x)\,dx}{\int_a^b X_n^2(x)\,dx}$$

<u>if</u> $X_n \perp X_m$ $\int_a^b X_n(x) X_m(x)\,dx = 0$, $n \neq m$.

<u>Theorem 5.3.1</u> If B.C. is symmetric, then any two eigenfunctions corresponding to different eigenvalues are orthogonal.

A B.C. is <u>symmetric</u> if for any two functions $f(x), g(x)$ satisfying the BC,

$$\left(f'(x)\overline{g(x)} - f(x)\overline{g'(x)} \right) \Big|_{x=a}^{x=b} = 0$$

Example of non-symmetric BC:

$$X(a) = X(b), \quad X'(a) = 2X'(b)$$

Theorem 5.3.2 If B.C is symmetric, then all eigenvalues are real numbers | P68

(we proved this in Lecture 10)

Theorem 5.3.3 If B.C is symmetric, and $f(x) f'(x) \Big|_{x=a}^{x=b}$ for any f satisfying the BC, then all eigenvalues ≥ 0.

Linear algebra $A = (a_{ij})_{n\times n}$ is a real-value $n\times n$ matrix

A is symmetric if $A = A^T$ (A^T = transpose of A) $a_{ij} = a_{ji}$

Thm If A is symmetric, then (i) eigenvectors for $\lambda_i \neq \lambda_j$ are orthogonal
(Hermitian for complex A) (ii) eigenvalues are real numbers.

$A = (a_{ij})_{n\times n}$ is complex-valued.

A is Hermitian if $A = \bar{A}^T$ ($a_{ij} = \overline{a_{ji}}$)

Thm If A is Hermitian, A is positive definite, then eigenvalue ≥ 0.

$x^T A x \geq 0, \ \forall x \in \mathbb{R}^n$,

Theorem 5.4.1 If B.C is symmetric There exist an infinite number of eigenvalues.

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \longrightarrow +\infty.$$

Convergence of Fourier series

Calc II $\sum_{n=1}^{\infty} a_n$ is defined by $\lim_{k\to\infty}\left(\sum_{n=1}^{k} a_n\right)$ if it exists. we say $\sum_{n=1}^{\infty} a_n$ is convergent

What about a function series? $\sum_{n=1}^{\infty} f_n(x)$, $x\in[a,b]$.

Math 311 (Elementary Analysis) Three kinds of convergences

① For every $x\in[a,b]$, $\lim_{k\to\infty}\sum_{n=1}^{k} f_n(x)$ exists

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \lim_{k\to\infty}\sum_{n=1}^{k} f_n(x)$$

we say $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ pointwisely

$$\lim_{k\to\infty}\left|\sum_{n=1}^{k} f_n(x) - f(x)\right| = 0, \quad \forall x\in[a,b]$$

② $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ uniformly for $x\in[a,b]$ if

$$\lim_{k\to\infty}\max_{a\le x\le b}\left|\sum_{n=1}^{k} f_n(x) - f(x)\right| = 0$$

③ $\sum_{n=1}^{\infty} f_n(x)$ Converges to $f(x)$ in the mean-square or L^2-Convergent if

$$\lim_{k\to\infty}\int_a^b\left|\sum_{n=1}^{k} f_n(x) - f(x)\right|^2 dx = 0.$$

Example $f_n(x) = x^{n-1} - x^n$

$$f(x) = \sum_{n=1}^{\infty} \left(x^{n-1} - x^n\right) = (1-x) + (x-x^2) + (x^2-x^3) + \cdots = 1$$

Partial sum $F_k(x) = \sum_{n=1}^{k} f_n(x) = 1 - x^k$

Does $F_k(x) = 1 - x^k$ converges to 1 in $0 \le x \le 1$?

Pointwise ✓ (~~ex~~ except at $x = 1$) $\lim_{k\to\infty} (1 - x^k) = 1$ if $|x| < 1$

(Sometimes say converge almost everywhere)

uniform ✗ $\max_{1 \le x \le 1} \left| (1-x^k) - 1 \right| = 1$ So $\lim_{k\to\infty} \neq 0$

What about $x \in (0,1)$?

Let $x_k = 1 - \frac{1}{k} = \frac{k-1}{k}$ Then $\left| (1 - x_k^{\,k}) - 1 \right| = \left| \left(1 - \frac{1}{k}\right)^k \right|$

$\lim_{k\to\infty} \left| \left(1 - \frac{1}{k}\right)^k \right| = \frac{1}{e} \neq 0$. So not uniform convergent in $(0,1)$.

L^2 ✓ $\int_0^1 \left| (1-x^k) - 1 \right|^2 dx = \int_0^1 x^{2k}\, dx = \frac{1}{2k+1} x^{2k+1} \Big|_0^1 = \frac{1}{2k+1} \to 0$

$(k \to \infty)$

In general $\boxed{\text{uniform convergence}} \Rightarrow \boxed{\text{pointwise convergence}}$

$\Rightarrow \boxed{L^2 \text{ convergence}}$

Function spaces

$C[a,b] = \{ u(x) : u(x) \text{ is continuous for every } x \in [a,b] \}$

$L^2(a,b) = \{ u(x) : \int_a^b |u(x)|^2 dx \text{ exists and is finite} \}$

$$L^2(a,b) \supseteq C[a,b]$$

Examples ① $f_1(x) = \begin{cases} -1 & 0 \le x \le \frac{1}{2} \\ 1 & \frac{1}{2} < x \le 1 \end{cases}$ $\quad f_1 \notin C[a,b],\ f_1 \in L^2(a,b)$

piecewise continuous

② $f_2(x) = \begin{cases} \frac{1}{\sqrt{x}} & 0 < x \le 1 \\ 2 & x = 0 \end{cases}$ $\quad f_2 \notin C[a,b],\ f_2 \in L^2(a,b)$

$$\int_0^1 \frac{1}{\sqrt{x}} dx = 2\sqrt{x}\Big|_0^1 = 2$$

(not continuous at $x=0$, $\lim_{x \to 0^+} f_2(x)$ does not exist)

Thm 5.4.3 If $f \in L^2(a,b)$, then the Fourier series of f converges to f in L^2-convergence.

Thm 5.4.4 ① If $f \in C[a,b]$ and f' is piecewise continuous on $[a,b]$, then the Fourier series of f converges to f for $\forall x \in (a,b)$.

(but maybe not at $x=a$, $x=b$!)

② If f, f' are piecewise continuous on $[a,b]$, then the Fourier series of f converges to $\frac{1}{2}[f(x+) + f(x-)]$, $\forall x \in (a,b)$

$$f(x+) = \lim_{y \to x^+} f(y) \qquad f(x-) = \lim_{y \to x^-} f(y)$$

Thm 5.4.3 If f, f', f'' are all continuous for $a \le x \le b$, and f satisfies the BC (the one satisfied by Fourier series), then the Fourier series converges to $f(x)$ uniformly in $[a, b]$.

p72

We have shown the Fourier sine series of $f(x) = 1$ on $(0, l)$ is

$$1 = \sum_{n\ \text{odd}} \frac{4}{n\pi} \sin(nx)$$

Let $x = \frac{l}{2}$ $\Rightarrow$ $1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ (Is this really true?) (Leibniz formula)

Thm 5.4.4 $f(x)$ is continuous, $f' = 0$ is continuous on $[0, l]$

So $\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges to 1 !

Fourier series and Taylor series $y = f(x)$ $\quad -l \le x \le l$

			Smoothness		
Fourier	$f(x) = \sum_{n=1}^{\infty} C_n X_n(x)$	$C_n = \frac{\int_{-l}^{l} f(x) X_n(x) dx}{\int_{-l}^{l} X_n^2(x) dx}$	$f \in L^2$	C_n is nonlocal	C_n is integral
Taylor	$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$	$C_n = f^{(n)}(0)$	$f \in C^\infty$ analytic	C_n is local	C_n is derivative

	convergence in L^2	converge pt. wise	converge uniformly	approximation
Fourier	always	mostly in $(-l, l)$	hard	good in L^2
Taylor	no	yes in $(-a, a)$ $0 < a < l$	yes in $(-a, a)$ $0 < a < l$	good if close to 0

$\rightarrow \frac{1}{a} = \limsup_{n \to \infty} \sqrt[n]{|C_n|}$ or $a = \lim_{n \to \infty} \left| \frac{C_n}{C_{n+1}} \right|$

other bases: Hermite Polynomials (orthogonal polynomials)
wavelets