Math 442 Lecture 13

$au_x + bu_y + cu = F(x,y)$ → more general ~~a~~ a, b, c are all functions of (x,y)

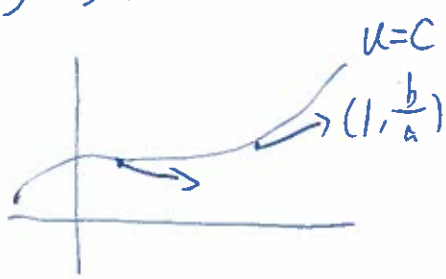
$a, b, c \in \mathbb{R}$

$$a(x,y)\,u_x + b(x,y)\,u_y + c(x,y)\,u = F(x,y).$$

<u>Simpler case</u> $a(x,y)u_x + b(x,y)u_y = 0$. or $u_x + \dfrac{b(x,y)}{a(x,y)}u_y = 0$.

$\Rightarrow$ directional derivative along $\left(1, \dfrac{b(x,y)}{a(x,y)}\right)$ is 0

$\Rightarrow$ Solution $u(x,y) = C$ if $y = y(x)$ is an orbit of $\dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)}$



(The solution curve $y = y(x)$ of $\dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)}$

$\Rightarrow$ directional derivative $= \dfrac{b(x,y)}{a(x,y)}$.

Suppose $y = y(x)$ satisfies $\dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)}$

Then $\dfrac{d}{dx}u(x, y(x)) = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}y'(x) = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\dfrac{b(x,y)}{a(x,y)} = u_x + u_y\dfrac{b(x,y)}{a(x,y)}$

$$= \frac{1}{a(x,y)}\left(a(x,y)u_x + b(x,y)u_y\right) = 0$$

Recipe 2 for $a(x,y)\,u_x + b(x,y)u_y = 0$

<u>Step 1</u> Solve characteristic curve $\dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)}$ $\Rightarrow$ solution $g(x,y) = C$

<u>Step 2</u> Solution $u(x,y) = f(g(x,y))$ f, g differentiable

Example 5 $\begin{cases} (1+x^2)u_x + u_y = 0 \\ u(1,y) = \cos y \end{cases}$

3.2 #2

$u_x + \frac{1}{1+x^2} u_y = 0 \Rightarrow$ Characteristic equation $\frac{dy}{dx} = \frac{1}{1+x^2}$

$\int dy = \int \frac{dx}{1+x^2} \Rightarrow y = \tan^{-1}(x) + C \quad y + C = \tan^{-1} x \Rightarrow \tan(y+C) = x.$

$\cancel{\tan(y+c)} \quad y - \tan^{-1} x = C \Rightarrow u(x,y) = f(y - \tan^{-1} x)$

$u(1,y) = \cos y = f(y - \tan^{-1} 1) = f(y - \frac{\pi}{4})$ Let $w = y - \frac{\pi}{4}$

$y = w + \frac{\pi}{4}$

$f(w) = \cos(w + \frac{\pi}{4}) \Rightarrow u(x,y) = \cos(y - \tan^{-1} x + \frac{\pi}{4})$

What about non-homogeneous case?

$a(x,y)u_x + b(x,y)u_y + c(x,y)u = F(x,y).$

Coordinate method

homogeneou case: $a(x,y)u_x + b(x,y)u_y = 0 \Rightarrow$ solution $g(x,y) = C$

$\begin{cases} W = g(x,y) \\ Z = x \text{ or } y \end{cases}$ (let's use $Z = y$)

$u_x = u_w w_x + u_z z_x = u_w \cdot g_x$

$u_y = u_w w_y + u_z w_y = u_w g_y + u_z$

So $a(x,y)(u_w \cdot g_x) + b(x,y)(u_w g_y + u_z) + c(x,y)u = F(x,y)$

$\Rightarrow u_w \underbrace{(a g_x + b g_y)}_{=0} + b(x,y)u_z + c(x,y)u = F(x,y)$

$\Rightarrow \tilde{b}(w,z)u_z + \tilde{c}(w,z)u = \tilde{F}(w,z)$

$g(x,y(x)) = C$

$\Rightarrow g_x + g_y \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{g_x}{g_y} = \frac{b(x,y)}{a(x,y)}$

Recipe 3 for $a(x,y)u_x + b(x,y)u_y + c(x,y)u = F(x,y)$ | P67

<u>Step 1</u> Solve characteric curve $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \Rightarrow$ Solution $g(x,y)=C$

<u>Step 2</u> Change coordinate $\begin{cases} w = g(x,y) \\ z = y \end{cases}$

<u>Step 3</u> Solve the new equation.

<u>Example 6</u> $\begin{cases} xu_x + yu_y + 2u = 0 \\ u(x,1) = \sin x \end{cases}$

<u>Step 1</u> $\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln y = \ln x + C \Rightarrow y = Cx$

or $\frac{y}{x} = C$ or $\frac{x}{y} = C$.

<u>Step 2</u> $\begin{cases} w = \frac{x}{y} \\ y = y \end{cases}$

$u_x = u_w w_x + u_y y_x = \frac{1}{y} u_w$

$u_y = u_w w_y + u_y y_y = -\frac{x}{y^2} u_w + u_y$

$\Rightarrow x \cdot \frac{1}{y} u_w + y\left(-\frac{x}{y^2} u_w + u_y\right) + 2u = 0$

$\boxed{(x,1) \to (yw,1) \to (w,1) \quad y=1 \text{ here}}$

$\begin{cases} yu_y + 2u = 0 \\ u(w,1) = \sin(w) \end{cases}$

$\frac{\partial u}{\partial y} = -\frac{2u}{y} \Rightarrow \frac{\partial u}{u} = -\frac{2\partial y}{y} \Rightarrow \ln u = -2\ln y + f(w)$

$\Rightarrow \ln(u) = \ln(y^{-2} f(w)) \Rightarrow u = \frac{1}{y^2} f(w) = \frac{1}{y^2} f\left(\frac{x}{y}\right)$

$u(x,1) = \sin x = f(x) \Rightarrow f(x) = \sin x \Rightarrow u(x,y) = \frac{1}{y^2} \sin\left(\frac{x}{y}\right)$

defined for $x \in \mathbb{R}$ and $y > 0$

Summary

linear 1st order PDE

$$a(x,y)\,u_x + b(x,y)\,u_y + c(x,y)\,u = F(x,y)$$

or even semilinear 1st order PDE

$$a(x,y)\,u_x + b(x,y)\,u_y = F(x,y,u)$$

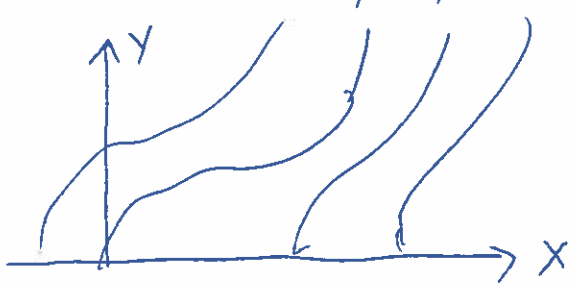
can be solved by the method of characteristic curves

That are solutions of ODE $\dfrac{dx}{a(x,y)} = \dfrac{dy}{b(x,y)}$ or $\dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)}$

$g(x,y) = c$

Then $\begin{cases} w = g(x,y) \\ y = y \end{cases}$ converts it to

$$u_y = \tilde{F}(x,y,u) \rightarrow \text{essentially an ODE}$$



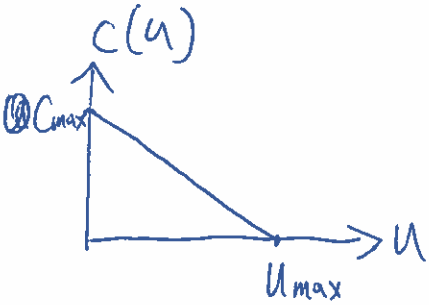
Uniqueness of solutions of ODE

$\Rightarrow$ characteristic curves do not cross each other.

Nonlinear 1st order PDE

① traffic flow model $\quad u(x,t)$ = density of cars at (x,t)

$u_t + c(u)\,u_x = 0 \Rightarrow$ velocity at car $= c(u)$ depending on density u.



u increases $\Rightarrow c(u)$ decreases

$$c(u) = C_{max}\left(1 - \frac{u}{u_{max}}\right)$$

② Burger's equation (gas dynamics)

$$\begin{cases} u_t + u\,u_x = 0 \\ u(x,0) = g(x) \end{cases}$$

Math 442 Lecture 3

Application of 1st order linear PDE

(2) Age-structure population model

$$\frac{du}{dt} = bu \quad \text{(Malthus model)} \qquad u(t) = \text{total population}$$

idea: divide the population according to age

$$u(n) = \begin{pmatrix} u_1(n) \\ u_2(n) \\ \vdots \\ u_K(n) \end{pmatrix} \qquad u_i(n) = \text{population of the } i\text{-th age group at time } n$$

$$u(n+1) = L \cdot u(n) \qquad L = \begin{pmatrix} s_1 m_1 & s_1 m_2 & \cdots & s_1 m_{K-1} & s_1 m_K \\ s_2 & 0 & \cdots & 0 & 0 \\ 0 & s_3 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & s_K & 0 \end{pmatrix}$$

Leslie matrix model (1945)

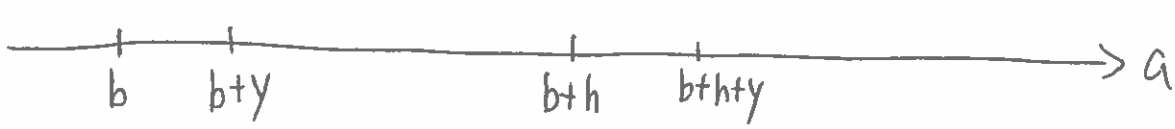
s_i = survival rate from age $i-1$ to i $\quad (0 < s_i < 1)$

m_i = reproduction rate per capita at age i $\quad m_i > 0$ for $\alpha \le i \le \beta$ fertile period

(Math 345) discrete time, discrete age structure

New idea: continuous time, continuous age structure

$u(t, a)$ = population density for age a at time t



$$\underbrace{\int_b^{b+y} u(t,a)\,da}_{\substack{\text{population between} \\ \text{age } b \text{ and } b+y \\ \text{at time } t}} = \underbrace{\int_{b+h}^{b+h+y} u(t+h, a)\,da}_{\substack{\text{population between} \\ \text{age } b+h \text{ and } b+h+y \\ \text{at time } t+h}} \qquad (\text{if no death!})$$

$$\int_b^{b+y} u(t,a)\,da = \int_{b+h}^{b+h+y} u(t+h,a)\,da$$

$\frac{d}{dy}$ $\quad u(t, b+y) = u(t+h, b+h+y)$

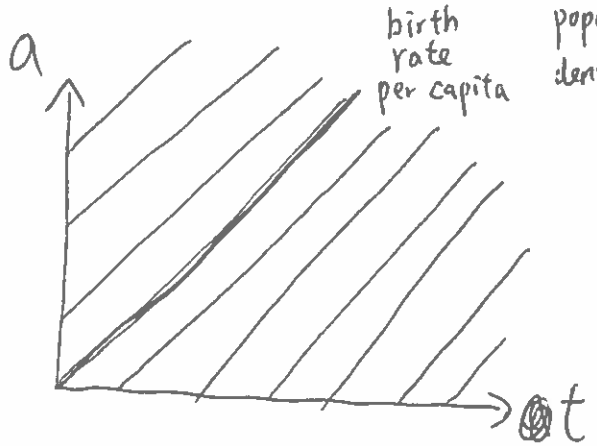
$\frac{d}{dh}$ $\quad 0 = u_t(t+h, b+h+y)\cdot 1 + u_a(t+h, b+h+y)\cdot 1$

$y=0$ and $h=0$ $\quad 0 = u_t(t,b) + u_a(t,b) = 0$

change $b \to a$ $\quad 0 = u_t(t,a) + u_a(t,a)$

Equation of aging

$$\begin{cases} u_t + u_a = 0, & t>0,\ a>0, \\ u(0,a) = f(a), & a>0 \quad \text{(initial distribution)} \\ u(t,0) = \int_0^\infty \underbrace{b(a)}_{\text{birth rate per capita}}\ \underbrace{u(t,a)}_{\text{population density}}\,da = \text{total new birth} & \text{(boundary condition)} \end{cases}$$



$a > t \Rightarrow$ determined by initial condition

$t > a \Rightarrow$ determined by "new" birth

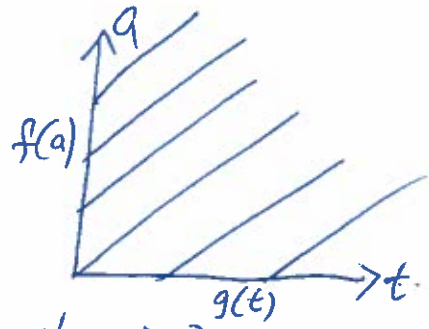
Mckendrick-Von Foerster Equation

Question: If for age a, the death rate is $d(a)$, then what will be the equation?

$$\int_b^{b+y} u(t,a)\,da = \int_{b+h}^{b+h+y} u(t+h,a)\,(1+d(a))\,da$$

A simpler version of age-structure model

$$\begin{cases} u_t + u_a = 0 & t>0,\ a>0 \\ u(0,a) = f(a), & a>0 \\ u(t,0) = g(t), & t>0 \end{cases}$$



$u(t,a) = h(t-a)$ when $t-a>0$

$\cancel{h(t-a) = g}$ $u(t,a) = u(t-a,0)$
$= g(t-a)$

when $t-a<0$ $u(t,a) = u(0,a-t) = f(a-t)$

$$\Rightarrow u(x,t) = \begin{cases} f(a-t) & \text{if } 0<t<a \\ g(t-a) & \text{if } t>a \end{cases}$$

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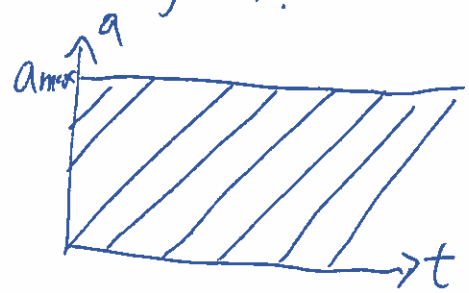
but nobody lives forever! More realistic model

$$\begin{cases} u_t + u_a + d(a)u = 0 \\ u(0,a) = f(a), & 0<a<a_{max} \\ u(t,0) = \int_0^{a_{max}} b(a)u(t,a)\,da \\ u(t,a_{max}) = 0 \end{cases}$$

$d(a)$ = death rate in age a.



characteristic curve $u_t + u_a = 0 \Rightarrow t-a = C$ $\quad w = t-a$ $\quad \begin{cases} w = t-a \\ a = a \end{cases}$

On the curve $u_w = -d(a)u$

$u(w,a) = u(0)e^{-d(a)t}$

For this model, $\lim_{t\to\infty} u(t,a) = u_d(a)$

a stable age distribution