# Math 442 Lecture 12



**Example** $\phi(x) = x$, $0<x<l$. Find its sine and cosine Fourier series.

$$x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \qquad A_n = \frac{2}{l}\int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx = \frac{2}{l}\cdot\frac{-l^2}{n\pi}(-1)^n = \frac{2l}{n\pi}(-1)^{n+1}$$

$$x = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) \qquad A_n = \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\cdot\frac{l^2}{n^2\pi^2}\left((-1)^n - 1\right) = \frac{2l}{n^2\pi^2}\left((-1)^n-1\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ -\dfrac{4l}{n^2\pi^2} & n \text{ odd} \end{cases} \qquad A_0 = \frac{2}{l}\int_0^l x\,dx = l$$

$$x = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2l}{n\pi}\sin\left(\frac{n\pi x}{l}\right), \qquad x = \frac{l}{2} + \sum_{n \text{ odd}} \frac{4l}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$$

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$$x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \frac{1}{4}\sin\frac{4\pi x}{l} + \cdots\right) \qquad \text{Sine series}$$

Let $x = \frac{l}{2}$ $\Rightarrow$ $\frac{l}{2} = \frac{2l}{\pi}\left(\sin\frac{\pi}{2} - \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} - \frac{1}{4}\sin 2\pi + \cdots\right)$

$$\frac{l}{2} = \frac{2l}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots\right)$$

Leibniz (1600–1700)
Madhava (1400)
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}\cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}$$

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$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \cdots\right)$$

Let $x = l$ $\Rightarrow$ $l = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\pi + \frac{1}{9}\cos 3\pi + \frac{1}{25}\cos 5\pi \cdots\right)$

$$\Rightarrow \frac{l}{2} = \frac{-4l}{\pi^2}\left(-1 - \frac{1}{9} - \frac{1}{25} - \frac{1}{49}\cdots\right)$$

$$\Rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Calculate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (we know it converges in Calc II)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n \text{ odd}} \frac{1}{n^2} + \sum_{n \text{ even}} \frac{1}{n^2} = \sum_{n \text{ odd}} \frac{1}{n^2} + \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) \cdot \frac{1}{4} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

So let $k = \sum_{n=1}^{\infty} \frac{1}{n^2}$ then $k = \frac{\pi^2}{8} + \frac{1}{4}k \Rightarrow \frac{3}{4}k = \frac{\pi^2}{8} \Rightarrow k = \frac{\pi^2}{6}$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{(Euler 1735)}$$

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3 types of Fourier series:

① Fourier Sine Series: $\phi(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$, $B_n = \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx$

② Fourier Cosine Series: $\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$ $A_n = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{n\pi x}{l}\right)dx$

Basis: Sine or Cosine functions with $\frac{1}{2}$ period $= l$.

③ Full Fourier series: $\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right)\right)$

Basis: both

$A_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\left(\frac{n\pi x}{l}\right)dx$, $B_n = \int_{-l}^{l} \phi(x) \sin\left(\frac{n\pi x}{l}\right)dx$

① comes from Dirichlet BC: $X(0) = X(l) = 0$

② ——— Neumann BC: $X'(0) = X'(l) = 0$

③ ——— periodic BC: $X(-l) = X(l)$
$X'(-l) = X(l)$

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Which one is better? $\phi(-x) = \phi(x)$ <u>even function</u>

$\phi(-x) = -\phi(x)$ <u>odd function</u>

Observations.

① If $\phi: [-l, l] \to \mathbb{R}$ is even, then $B_n = 0$ for all $n$. (Cosine series)

② If $\phi: [-l, l] \to \mathbb{R}$ is odd, then $A_n = 0$ for all $n$. (Sine series)

③ Any $\phi: [-l, l] \to \mathbb{R}$ can be written as

$$\phi(x) = \phi_{even}(x) + \phi_{odd}(x) = \frac{1}{2}[\phi(x) + \phi(-x)] + \frac{1}{2}[\phi(x) - \phi(-x)]$$

$$= \underbrace{\left(\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)\right)}_{\text{even}} + \underbrace{\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)}_{\text{odd}}.$$

④ $\phi: [0, l]$ can be extended to $\phi_E: [-l, l]$ as an even function

$\phi:$ ———————— $\phi_O: [-l, l]$ as an odd function

Cosine series of $\phi$ = Full series of $\phi_E$.

Sine series of $\phi$ = Full series of $\phi_O$.

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Type ④ (If you do not mind complex numbers)

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

$$\boxed{\begin{aligned} &\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \text{(De Moivre's formula)} \\ &e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{-i\theta} = \cos\theta - i\sin\theta \qquad \text{(Euler's formula)} \end{aligned}}$$

$$= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\left(\frac{e^{i\frac{n\pi x}{l}} + e^{-i\frac{n\pi x}{l}}}{2}\right) + B_n \frac{e^{i\frac{n\pi x}{l}} - e^{-i\frac{n\pi x}{l}}}{2i}$$

$$= \frac{1}{2}\left[A_0 + \sum_{n=1}^{\infty}(A_n - B_n i)e^{i\frac{n\pi x}{l}} + \sum_{n=1}^{\infty}(A_n + B_n i)e^{-i\frac{n\pi x}{l}}\right.$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{l}} \qquad C_n = \frac{1}{2l}\int_{-l}^{l} \phi(x) e^{-i\frac{n\pi x}{l}} dx$$

$$C_n = \overline{C_{-n}}$$

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Summary

① For a continuous function $\phi(x)$ on $[0,l]$, (or $[-l,l]$), Fourier series can be defined. This can help to solve PDEs like diffusion Eq, Wave Eq.

② Choice of Fourier series depends on BC.

③ Mathematical question

$$\phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{l}x\right)$$

What does "=" mean? (Convergence Question)

series $$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{l}x\right) = \lim_{K\to\infty} \sum_{n=1}^{K} C_n \sin\left(\frac{n\pi}{l}x\right)$$

Does the limit really exist? If so, in what sense?