Math 442 Lecture 11

Review of linear algebra

$\mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$ basis of $\mathbb{R}^2$: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

$$\forall \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}, \quad \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{a+5b}{11} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{2a-b}{11} \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

basis : a set of vectors so that (1) any vector can be generated by a linear combination of vectors in basis ; (2) linear independent.

dimension of $\mathbb{R}^2$ = elements of vectors in a basis = 2.

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is an orthonormal basis as $\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = 0$

$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = x_1 x_2 + y_1 y_2$ is the dot product in $\mathbb{R}^2$

inner product

Same for $\mathbb{R}^n$, $\dim(\mathbb{R}^n) = n$, Any finite dimensional vector space is "same as" $\mathbb{R}^n$.

Space of functions

$C([a,b])$ = set of continuous functions on interval $[a,b]$.

dimension of $C([a,b]) = \infty$

(1) $1, x, x^2, x^3, x^4, \cdots$ are linear independent

Suppose that $a_0 \cdot 1 + a_1 \cdot x + a_2 x^2 + \cdots + a_n x^n = 0$

The polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$ has only n roots.

Let $x \neq x_i$ ($i=1, \cdots n$) roots of $p(x)=0$

Then $p(x) \neq 0 \Rightarrow$ not linear dependent. $\Rightarrow \dim = \infty$.

Is $\{x^i : i \in \mathbb{N} \cup \{0\}\}$ a basis of $C([a,b])$?

$f(x) = \sum_{i=0}^{\infty} a_i x^i$ power series : may not converge

orthonormal ? inner product in $C([a,b])$

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

Question : ① Is there a basis for $C([a,b])$?

② Is there an orthonormal basis ?

We will work on $[a,b] = [0, \ell]$, $\ell > 0$

Define $S = \{\sin(\frac{n\pi x}{\ell}) ; n \in \mathbb{N}\}$, $C = \{\cos(\frac{n\pi x}{\ell}) ; n \in \mathbb{N} \cup \{0\}\}$.

① S is an orthogonormal set.

$$\int_0^\ell \sin(\frac{n\pi}{\ell}x) \sin(\frac{m\pi}{\ell}x)\, dx = \frac{1}{2}\int_0^\ell \left[\cos\frac{(m-n)\pi x}{\ell} - \cos\frac{(m+n)\pi x}{\ell}\right] dx$$

$$= \left(\frac{\ell}{2(m-n)\pi} \sin\frac{(m-n)\pi x}{\ell} - \frac{\ell}{2(m+n)\pi}\sin\frac{(m+n)\pi x}{\ell}\right)\Bigg|_0^\ell = 0 \quad (\text{if } m \neq n)$$

if $m=n$

$$= \frac{1}{2}\int_0^\ell \left[1 - \cos\frac{2m\pi x}{\ell}\right] dx = \frac{\ell}{2} - \frac{\ell}{4m\pi}\sin\frac{2m\pi x}{\ell}\Bigg|_0^\ell = \frac{\ell}{2}$$

② Elements in S are linear independent

Suppose $\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) = 0$, multiply $\sin\left(\frac{m\pi x}{l}\right)$ and integrate,

$$\sum_{n=1}^{\infty} a_n \int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right) dx = 0 \Rightarrow$$ all integrals $= 0$ exapt $n = m$

$$\Rightarrow a_m \cdot \int_0^l \left[\sin\left(\frac{m\pi x}{l}\right)\right]^2 dx = 0 \Rightarrow a_m \cdot \frac{1}{2} l = 0 \Rightarrow a_m = 0.$$

m is arbitrary $\Rightarrow a_n = 0 \ \forall n \in \mathbb{N} \Rightarrow$ linearly independent.

③ S can generate $C([0,l])$

Let $\phi(x) \in C([0,l])$. Can we find C_n so that $\phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{l}x\right)$?

multiply $\sin\left(\frac{m\pi x}{l}\right)$ and integrate!

$$\int_0^l \phi(x)\sin\left(\frac{m\pi x}{l}\right) dx = \sum_{n=1}^{\infty} C_n \int_0^l \sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right) dx = C_m \cdot \frac{l}{2}$$

So $$C_m = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{m\pi x}{l}\right) dx.$$

"Theorem" S is a basis of $C([0,l])$. For $\forall \phi \in C([0,l])$

let $$C_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right) dx$$ (Fourier sine series of $\phi(x)$)

then $$\phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

~~So is C, $\forall \phi \in C([0,l])$, $\phi(x) = \Sigma$~~

Compeletion of Solutions of wave / diffasion equations

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0<x<l \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x) \end{cases}$$

Series solution

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

$$u_t(x,t) = \sum_{n=1}^{\infty}\left(-A_n \sin\left(\frac{n\pi ct}{l}\right) + B_n\cos\frac{n\pi ct}{l}\right)\cdot\frac{n\pi c}{l}\sin\left(\frac{n\pi x}{l}\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = \phi(x)$$

$$u_t(x,0) = \frac{n\pi c}{l}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \psi(x)$$

$$\begin{cases} u_t = k u_{xx}, & 0<x<l \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Series solution

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n\pi}{l}\right)^2 t}\sin\left(\frac{n\pi}{l}x\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{l}x\right) = \phi(x)$$

$$C_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi}{l}x\right)dx$$

Solved !!

$$A_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$B_n = \frac{2}{l}\cdot\frac{l}{n\pi c}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{n\pi c}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

So initial position determines A_n, initial velocity determines B_n.

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wave equation $\Rightarrow$ time periodic

diffasion equation $\lim_{t\to\infty} u(x,t) = 0$. (decay to 0)

Neumann BC

$$\begin{cases} u_t = k u_{xx} & 0<x<l \\ u_x(0,t) = u_x(l,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

$$u(x,t) = \sum_{n=0}^{\infty} C_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} \cos\left(\frac{n\pi}{l}x\right)$$

$$u(x,0) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{l}x\right) = \phi(x) \qquad C_n = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{n\pi}{l}x\right)dx$$

$$C_0 \cdot \int_0^l 1 \cdot dx = \int_0^l \phi(x)\,dx$$

So the C_0 is different $\quad C_0 = \frac{1}{l}\int_0^l \phi(x)\,dx$

Usually we write

$$u(x,t) = \frac{1}{2}C_0 + \sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} \cos\left(\frac{n\pi}{l}x\right)$$

where $\quad C_n = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{n\pi}{l}x\right)dx\ , \quad n=0,1,2\cdots$

$$\lim_{t\to\infty} u(x,t) = \frac{1}{2}C_0 = \frac{1}{l}\int_0^l \phi(x)\,dx = \text{average initial value}$$

(recall Homework 2 #3) steady state solution

$\mathcal{C} = \left\{ \cos\left(\frac{n\pi}{l}x\right) ; n \in \mathbb{N} \cup \{0\} \right\}$ is also a basis of $C([0,l])$

$\forall \phi \in C([0,l])$, $\quad \phi(x) = \frac{1}{2}C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{l}x\right)$ (Fourier cosine series of $\phi(x)$)

$$C_n = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{n\pi}{l}x\right)dx$$

Example $\phi(x) = 1$

① $\phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{l}x\right)$

$$C_n = \frac{2}{l}\int_0^l 1 \cdot \sin\left(\frac{n\pi x}{l}\right)dx = \frac{-2}{l}\cdot\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\Big|_0^l$$

$$= -\frac{2}{n\pi}\left(\cos n\pi - 1\right) = -\frac{2}{n\pi}\left((-1)^n - 1\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$

So $\cancel{\phi(x)}$ $1 = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-(-1)^n\right)\sin\left(\frac{n\pi}{l}x\right)$

$$= \frac{4}{\pi}\sin\left(\frac{\pi x}{l}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi x}{l}\right) + \cdots$$

(Fourier sine series of $\phi(x)=1$)

② $\phi(x) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{l}x\right) + \frac{1}{2}C_0$

$$C_n = \frac{2}{l}\int_0^l 1 \cdot \cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\cdot\frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)\Big|_0^l$$

$$= \frac{2}{n\pi}\left(\sin(n\pi) - 0\right) = 0 \quad (n \geq 1)$$

$$C_0 = \frac{2}{l}\int_0^l 1 \cdot 1\, dx = 2$$

$\Rightarrow$ $\phi(x) = 1$ is the Fourier cosine series.

(of course!)

$$1 = \frac{4}{\pi}\left[\sin\left(\frac{\pi x}{l}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{l}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{l}\right) + \cdots\right]$$

Let $x = \frac{l}{2}$ $\quad 1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}\right)$

So $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1} = \frac{\pi}{4}$

(Gottfried Leibniz, 1675)

$(\tan^{-1}x)' = \frac{1}{1+x^2}$ $\qquad \int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ $\qquad \int_0^1 \sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$