Math 442 Lecture 10.

Separation of variables

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0<x<l, \\ u(0,t) = u(l,t) = 0 \end{cases} \qquad \begin{cases} u_t = k u_{xx}, & 0<x<l \\ u(0,t) = u(l,t) = 0. \end{cases}$$

Eigenvalue problem $\begin{cases} X'' = -\lambda X, & 0<x<l \\ X(0) = X(l) = 0 \end{cases}$

~~Solution~~ Eigenvalue: $\lambda_n = \left(\frac{n\pi}{l}\right)^2$, Eigenfunction: $X_n(x) = \sin\left(\frac{n\pi}{l}x\right)$

$n = 1, 2, 3, \cdots$

Solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_{3n} \cos\left(\frac{cn\pi}{l}t\right) + C_{4n} \sin\left(\frac{cn\pi}{l}t\right) \right) \sin\left(\frac{n\pi}{l}x\right) \qquad \underline{\text{wave}}$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} \sin\left(\frac{n\pi}{l}x\right)$$

It also works for other BCs.

The difference is on the eigenvalue problems

general eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 & 0<x<l \\ \text{Boundary conditions} \end{cases}$$

Eigenvalues
1. positive : most common
2. zero
3. negative
4. complex

Lemma (Thm 5.2–5.3) If the boundary condition ~~makes~~ satisfies $X'X \leq 0$ on boundary, then λ is real and $\lambda \geq 0$ (no ③ or ④)

proof. First we claim that if $\lambda = \alpha + i\beta$ is an eigenvalue, then $\bar{\lambda} = \alpha - i\beta$ is also an eigenvalue.

$X = u + iv \quad 0 = X'' + (\alpha + i\beta)X = (u+iv)'' + (\alpha+i\beta)(u+iv)$

$$= (u'' + \alpha u - \beta v) + i(v'' + \beta u + \alpha v) = 0$$

So $u'' + \alpha u - \beta v = 0$ and $v'' + \beta u + \alpha v = 0$.

Let $\bar{X} = u - iv \quad \bar{X}'' + \overline{(\alpha+i\beta)}\bar{X} = (u-iv)'' + (\alpha - i\beta)(u - iv)$

$$= (u'' + \alpha u - \beta v) - i(v'' + \beta u + \alpha v)$$

So $\bar{X}'' + \overline{(\alpha+i\beta)}\bar{X} = 0$

Multiplying $X'' + \lambda X = 0$ by $\bar{X}$, and integrating on $(0, l)$

$$0 = \int_0^l (X'' \cdot \bar{X} + \lambda X \cdot \bar{X})\, dx = -\int_0^l X' \cdot \bar{X}' + X'\bar{X}\Big|_0^l + \lambda \int_0^l X \cdot \bar{X}$$

$$= -\int_0^l X' \cdot \bar{X}' + \lambda \int_0^l X \cdot \bar{X} \Rightarrow \lambda = \frac{\int_0^l X' \cdot \bar{X}'\, dx}{\int_0^l X \cdot \bar{X}\, dx}$$

$X = u + iv \qquad X' = u' + iv'$

So $X' \cdot \bar{X}' = (u' + iv') \cdot (u' - iv') = (u')^2 + (v')^2$

$X \cdot \bar{X} = (u + iv)(u - iv) = u^2 + v^2$

So $$\lambda = \frac{\int_0^l [(u')^2 + (v')^2]\, dx \longrightarrow \text{real value}}{\int_0^l [u^2 + v^2]\, dx \longrightarrow \text{real value}} \geq 0.$$

$\blacksquare$

Dirichlet BC $\qquad \lambda_n = \left(\frac{k\pi}{l}\right)^2 \quad n = 1, 2, \cdots$ all positive.

Neumann BC

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < l \\ X'(0) = X'(l) = 0 \end{cases}$$

From Lemma $\lambda \in \mathbb{R}$ and $\lambda \geq 0$.

Case 1 $\quad \lambda = 0$, $X(x) = ax + b \qquad X'(x) = a = 0 \Rightarrow X(x) = b$

It is okay! $\quad X(x) = 1$ and $\lambda = 0$

Case 2 $\quad \lambda > 0$

$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$X'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$X'(0) = C_2\sqrt{\lambda} = 0 \Rightarrow C_2 = 0$

$X'(l) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}l = 0 \Rightarrow \sin\sqrt{\lambda}l = 0$

$\Rightarrow \sqrt{\lambda}l = n\pi, \quad n = 1, 2, \cdots$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad X_n(x) = \cos\left(\frac{n\pi}{l}x\right)$

Eigenvalues $\quad \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = \underline{0}, 1, 2, \cdots \quad X_n(x) = \cos\left(\frac{n\pi}{l}x\right)$

Other BCs

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X'(l) = 0 \end{cases}$$ mixed

Lemma $\Rightarrow \lambda \in \mathbb{R},\ \lambda \geq 0$ since $X \cdot X' \leq 0$ at $x=0$ or $x=l$.

Case 1 $\lambda = 0$ $\quad X(x) = c_1 x + c_2 \quad X'(x) = c_1$

$X'(l) = c_1 = 0 \Rightarrow X(x) = c_2$ but $X(0) = 0 = c_2$

$\Rightarrow c_1 = c_2 = 0$ no zero eigenvalue.

Case 2 $\lambda > 0$. $\quad X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$

$X(0) = c_1 \cdot 1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0$

$X'(x) = c_2 \sqrt{\lambda} \cos\sqrt{\lambda}\, x$

$X'(l) = c_2 \sqrt{\lambda} \cos\sqrt{\lambda} l = 0 \Rightarrow \cos\sqrt{\lambda} l = 0$

$$\Rightarrow \sqrt{\lambda} l = \left(n+\tfrac{1}{2}\right)\pi \Rightarrow \lambda_n = \frac{(n+\frac{1}{2})^2\pi^2}{l^2} = \frac{(2n+1)^2\pi^2}{4l^2}, \quad n = 1, 2 \cdots$$

$$X_n(x) = \sin\left(\frac{(2n+1)\pi x}{2l}\right)$$

Robin BC

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < l \\ X'(0) - a_0 X(0) = 0 \\ X'(l) + a_l X(l) = 0 \end{cases}$$

outer normal $\quad -X'(0) = -a_0 X(0), \quad X'(l) = -a_l X(l)$

radiation : $a_0 > 0,\ a_l > 0$ (leaking)

absorption : $a_0 < 0,\ a_l < 0$ (absorbing)

insulation : $a_0 = a_l = 0$ (Neumann, no flux)

Eigenvalues for Robin:

	$\lambda>0$	$\lambda=0$	$\lambda<0$
radiation	✓	✗	✗
absorption	✓	✓	maybe ✓
insulation	✓	✓	✗

$\lambda > 0$ $\quad X(x) = c_1 \cos\beta x + c_2 \sin\beta x \qquad \sqrt{\lambda} = \beta.$

$$X'(x) = -c_1\beta \sin\beta x + c_2 \beta \cos\beta x$$

$$0 = X'(0) - a_0 X(0) = c_2\beta - a_0 c_1 = 0$$

$$0 = X'(l) + a_l X(l) = (-c_1\beta\sin\beta l + c_2\beta\cos\beta l) + a_l(c_1\cos\beta l + c_2\sin\beta l)$$

$$A \equiv \begin{pmatrix} -a_0 & \beta \\ a_l\cos\beta l - \beta\sin\beta l & a_l\sin\beta l + \beta\cos\beta l \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

Solvable if $\det(A) = 0$.

$$-a_0 a_l \sin\beta l - a_0\beta\cos\beta l - \beta a_l \cos\beta l + \beta^2 \sin\beta l = 0. \quad \left(\text{divide by } \cos\beta l\right)$$

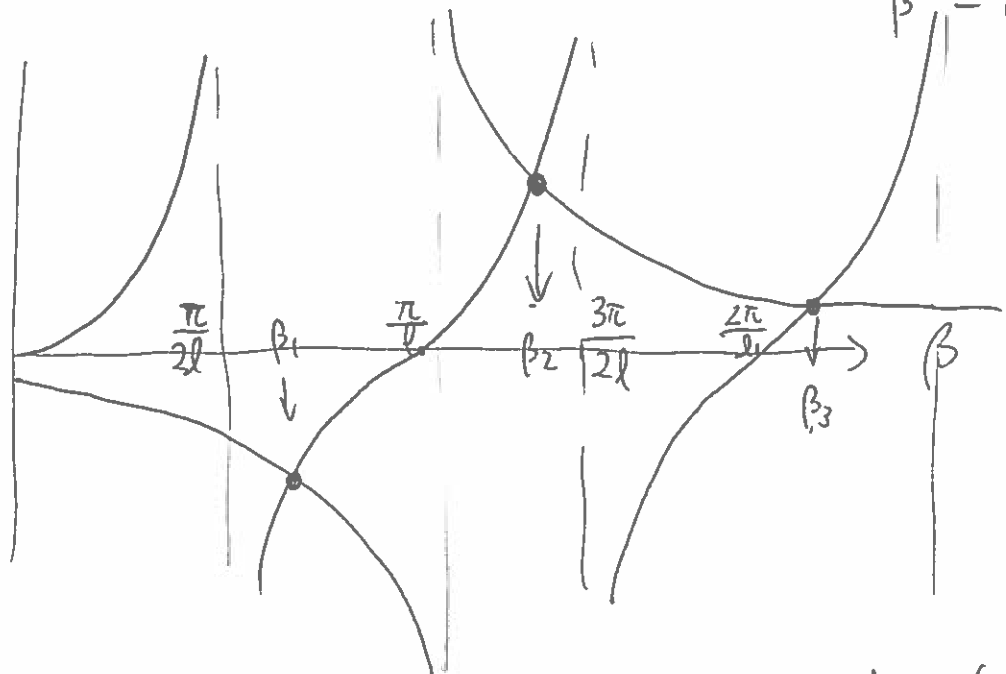
$$-a_0 a_l \tan\beta l - a_0\beta - a_l\beta + \beta^2\tan\beta l = 0$$

$$(\beta^2 - a_0 a_l)\tan\beta l = (a_0 + a_l)\beta$$

$$\tan\beta l = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l}$$

The eigenvalues are not explicitly solvable but $\beta_n = \sqrt{\lambda_n}$ is the n-th intersection of $f_1(\beta)$ and $f_2(\beta)$

$$f_1(\beta) = \tan(\beta \ell) \qquad f_2(\beta) = \frac{(a_0 + a_\ell)\beta}{\beta^2 - a_0 a_\ell}$$



$$X_n \doteq \beta_n \cos \beta_n x + a_0 \sin \beta_n x \qquad \lim_{n\to\infty} \left(\beta_n - \frac{n\pi}{\ell}\right)^0 = 0$$

So asymptotically $\lambda_n \sim \left(\frac{n\pi}{\ell}\right)^2$

<u>negative eigenvalues for absorption case</u> : see book .