Math 442 Lecture 1

What is a PDE (Partial differential equation)?

Equations of quantity of a substance depending on more than one independent variables

independent variables : t (time), (x,y,z) space $\mathbb{R}^3$,

biology : phenotype x, finance: price of stock S

dependent variable $u(\underbrace{t}_{\text{time}}, \underbrace{x,y,z}_{\text{location}})$

temperature at time t, location (x,y,z)

population density of time t, location (x,y) with phenotype p

$u(t,x,y,p)$

price to option of time t, stock price k

$V(t,k)$

partial derivative : rate of change of u with respect to one of independent variable

$$\frac{\partial u(t,x,y,z)}{\partial t} = \lim_{h\to 0} \frac{u(t+h,x,y,z) - u(t,x,y,z)}{h}$$

Notation : $\frac{\partial u}{\partial t}$, u_t, $\frac{\partial u}{\partial x}$, u_x, $\frac{\partial^2 u}{\partial x^2}$, u_{xx} ...

A PDE is an identity that relates all independent varibles, (t,x,y,z), dependent variable u, and its partial derivatives $(u_t, u_x, u_y, u_z, u_{xx}, \dots)$

$$F(t,x,u,u_t,u_x,u_{xx}) = 0$$

Order of PDE : the highest degree of derivative in the PDE

First order PDE : only have 1st order derivatives $u_t + u_x^2 = 0$

Second order PDE : has first and second order derivatives. $u_t + u_{xx} = 0$

A solution of a PDE is a function u depending on all independent variables, and it satisfies the equation.

Example 1 $\quad u_t = 1 \quad$ solution $u(x,t)$ $\quad \int u_t \, dt = \int 1 \, dt \Rightarrow u = t + C(x)$

$u(x,t) = C(x) + t$ satisfies $\frac{\partial u}{\partial t} = 1$ and $C(x)$ is an arbitrary function

Compare to ODE $\quad u_t = 1 \quad$ solution $u(t)$

Solution $u(t) = t + C$ where C is a constant

Example 2 $\quad$ Newton's cooling law $\quad \frac{dT}{dt} = -k(T - T_0)$ $\quad T_0$ = equilibrium temperature

$$T(t) = T_0 + (T(0) - T_0)e^{-kt}$$

Now assume $T(x,t)$ is the temperature on a line segment $0 \le x \le \pi$

$$(*) \quad \begin{cases} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} - T, & 0 < x < \pi \\ T(0,t) = T(\pi,t) = 0 \\ T(x,0) = \sin x \end{cases}$$

Verify $T(x,t) = e^{-2t} \sin x$ is a solution of $(*)$

$$\frac{\partial T}{\partial t} = -2e^{-2t} \sin x \qquad \frac{\partial^2 T}{\partial x^2} = (\sin x)_{xx} \, e^{-2t} = -\sin x \, e^{-2t}$$

So $LHS = -2e^{-2t} \sin x \qquad RHS = \frac{\partial^2 T}{\partial x^2} - T = -\sin x e^{-2t} - e^{-2t} \sin x = -2e^{-2t} \sin x$

Thus $LHS = RHS \Rightarrow$ it is a solution.

① This equation describes the diffusion of temperature (or heat) in $\frac{\partial^2 T}{\partial x^2}$

② $T(x,0) = \sin x$ is an initial condition. (same as ODE)

③ $T(0,t) = 0$ and $T(\pi, t) = 0$ are boundary conditions (new)

④ Another solution (not satisfying initial condition) $T_1(x,t) = e^{-5t}\sin(2x)$

⑤ Another solution (not satisfying BC) ?

A PDE ~~usually~~ problem usually has 3 parts

$\left\{\begin{array}{l}\text{Equation} \\ \text{Initial Condition} \\ \text{boundary Condition}\end{array}\right. \Rightarrow$ initial-boundary-value problem.

In general, a PDE can have infinitely many solutions without one of initial condition and boundary condition.

$$T_t = T_{xx} - T \quad \Rightarrow \quad T(x,t) = e^{-(k^2+1)t}\sin(kx), \quad \underbrace{k>0}_{\text{any positive}}$$

A PDE with proper IC and BC may have a unique solution.

A PDE with some IC and BC is well-posed if

1) it has a solution. (existence)

2) there is only one solution (uniqueness)

3) the ~~problem~~ solution varies continuously w.r.t IC and BC (stability)

The well-posedness could be a hard question for some PDEs.

One of millennium problems: Well-posed-ness of Navier–Stokes Equation ?

Same Question for ODE : Equation + initial condition

Theorem (Math 302): $x' = f(x,t)$, $x(0) = x_0$, $x(t)$

If f and f_x, f_t are continuous, then this ODE is well-posed for $t \in (-\delta, \delta)$.

Even the ODE may not be well-posed for $t\in[0,\infty)$

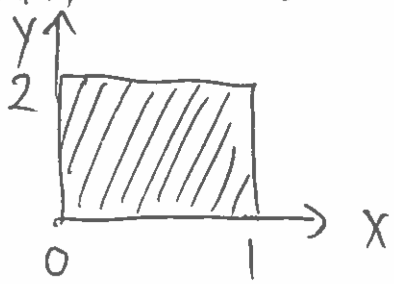
(a) $x' = x^2$, $x(0)=1$ $\quad x(t)=\frac{1}{1-t}$ $\quad t\in[0,1)$ <u>blow-up</u>

(b) $x'=\sqrt{x}$, $x(0)=0$ $\quad x(t)=0$ and $x(t)=\frac{1}{4}t^2$ <u>not unique</u>

(c) Lorentz equation for $x\in\mathbb{R}^3$ not continuous in IC for $t\in[0,\infty)$ <u>chaos</u>

<u>Example 3</u> (page 12 #7) Explain $u_{xx}+u_{yy}=0$, $u_x(0,y)=u_x(1,y)=u_y(x,0)=u_y(x,2)=0$

$0\le x\le 1$, $0\le y\le 2$



Let $u(x,y)=k$ for any $k\in\mathbb{R}$

Then it is a solution. So the solution is not unique, thus <u>not well-posed</u>.

(Easily) Solvable PDE

When there is only one partial derivative in the PDE.

<u>Example 4</u> $\quad u_{xyy} = x+y \quad u(x,y)$

$$\int u_{xyy}\,dx = \int (x+y)\,dx \Rightarrow u_{yy} = \frac{1}{2}x^2 + xy + \underline{f(y)} \rightarrow y \text{ is a constant when integrating in } x.$$

$$\int u_{yy}\,dy = \int\left(\frac{1}{2}x^2+xy+f(y)\right)dy \Rightarrow u_y = \frac{1}{2}x^2y+\frac{1}{2}xy^2+\int f(y)\,dy + g(x)$$

$$\int u_y\,dy = \int\left(\frac{1}{2}x^2y+\frac{1}{2}xy^2+f(y)+g(x)\right)dy = \frac{1}{4}x^2y^2+\frac{1}{6}xy^3+f(y)+g(x)y + h(x)$$

$$\text{So } u(x,y) = \frac{1}{4}x^2y^2+\frac{1}{6}xy^3 \cancel{+f(x)+g(y)+h} + h(x)+f(y)+g(x)y$$

3rd equation $\longrightarrow$ 3 arbitrary functions to choose.

Any PDE can be written in a form

$$L(u) = f(x,t)$$

$L(u)$ $\downarrow$ all terms with u and derivatives

$f(x,t)$ $\downarrow$ all terms without u or derivatives

If $f(x,t) = 0$, then the PDE is <u>homogeneous</u>

If $f(x,t) \neq 0$, then the PDE is <u>non-homogeneous</u> (same for ODE)

The PDE is <u>linear</u> if (1) $L(u_1 + u_2) = L(u_1) + L(u_2)$ (same for ODE)

(2) $L(cu) = cLu$ for $c \in \mathbb{R}$

otherwise it is <u>nonlinear</u>

<u>Example 5</u> Determine the PDEs are homogeneous/non-homogeneous, linear/nonlinear, 1st/2nd order

$u(x,t)$

$u_x + \sin x\, u_t + t^2 = 0$	NH, L, 1st
$u_x + \sin u \cdot u_t = 0$	H, NL, 1st
$u_{xx} + u_{tt} - u^2 = 0$	H, NL, 2nd
$\dfrac{u_x}{\sqrt{1+u_x^2}} = \sin t$	NH, NL, 1st

(H) $L(u) = 0$

(N-H) $L(u) = f(x,t)$

Linear Principle (Principle of superposition) for (H) if <u>L is linear</u>

① If u is a solution of (H), so is cu for $c \in \mathbb{R}$.

② If u_1 and u_2 are solutions of (H), so is $u_1 + u_2$.

③ If for $i \in \mathbb{N}$, u_i is a solution of (H), $c_i \in \mathbb{R}$, then

$\sum_{i=1}^{\infty} c_i u_i$ is also a solution of (H) if the sum converges.

Principle of Superposition for (NH) if $\underline{L \text{ is linear}}$

① If $L(u_1) = f$, $L(u_2) = g$, then $L(u_1 \pm u_2) = f \pm g$

In particular if $L(u_1) = 0$, $L(u_2) = f \Rightarrow L(u_1 + u_2) = f$

$L(u_1) = f$ and $L(u_2) = f \Rightarrow L(u_1 - u_2) = 0$

The Solutions of (H) is a $\underline{\text{linear space}}$ S

The solutions of (N-H) is $p(x,t) + h(x,t)$ where $h(x,t) \in S$

and $p(x,t)$ is any Solution of (N-H) $\underline{\text{(particular solution)}}$

$\underline{\text{Example 6}}$ ODE $u'' + u = 0$ (H) $\Rightarrow u(t) = C_1 \cos t + C_2 \sin t$

$u'' + u = 3$ (N-H) $\Rightarrow u(t) = \underbrace{3}_{\text{particular}} + \underbrace{c_1 \cos t + c_2 \sin t}_{\text{Solution of (H)}}$

$\dim(S) = 2$ (2-D linear space)

$S = \{ c_1 \cos t + C_2 \sin t : C_1, C_2 \in \mathbb{R} \}$

PDE $u_{xy} = 0$ (H) $u(x,y) = f(x) + g(y)$

$u_{xy} = 5$ (N-H) $u(x,y) = \underbrace{5xy}_{\text{particular}} + \underbrace{f(x) + g(y)}_{\text{Solution of (H)}}$

$S = \{ f(x) + g(y) : f \text{ and } g \text{ are differentiable} \}$.

$\dim(S) = \infty$ $1, x, x^2, \cdots x^n \cdots \in S$, $y, y^2, \cdots y^n \cdots \in S$

In this class,			
	1st order ✓ (later)	homogeneous ✓	linear ✓
	2nd order ✓	non-homogeneous (a little)	nonlinear (a little)