Math 442 Homework 5: (due February 23, 2018)

1. Page 89 (2) Consider a metal rod ($0 < x < l$), insulated along its side but not at its ends, which is initially at temperature= 1. Suddenly both ends are plunged into a bath of temperature= 0.

   (a) Write the differential equation, boundary conditions, and initial conditions.

   (b) Write the formula for the temperature $u(x,t)$ at later times. Here we assume that

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right) = \frac{4}{\pi}\sum_{i=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{l}.$$

2. Page 89 (3) A quantum-mechanical particle on the line with an infinite potential outside the interval $(0,l)$ ("particle in a box") is given by Schrödinger's equation $u_t = iu_{xx}$ on $(0,l)$ with Dirichlet conditions at the ends. Separate the variables and use Page 85 (8) to find its representation as a series.

3. Page 92 (2) Consider the boundary value problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < l, 0 < t < \infty, \\ u_x(0,t) = 0, \quad u(l,t) = 0, & t > 0. \end{cases}$$

   (a) Show that the eigenfunctions are $\cos[(n + 1/2)\pi x/l]$, and the eigenvalues are $(n + 1/2)^2\pi^2/l^2$.

   (b) Write the series expansion for a solution $u(x,t)$.

4. Page 92 (4) Consider the concentration of the diffusing substance satisfies

$$\begin{cases} u_t = ku_{xx}, & -l < x < l, 0 < t < \infty, \\ u(-l,t) = u(l,t), \quad u_x(-l,t) = u_x(l,t), & t > 0. \end{cases}$$

   These are called periodic boundary conditions.

   (a) Show that the eigenfunctions are $\cos(n\pi x/l)$ and $\sin(n\pi x/l)$, and the eigenvalues are $n^2\pi^2/l^2$ for $n = 0, 1, 2, 3, \cdots$.

   (b) Show that the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{l} + B_n\sin\frac{n\pi x}{l}\right)e^{-n^2\pi^2 kt/l^2}.$$