Math 442 Homework 4: (due February 16, 2018)

1. Page 45 (1) Consider the solution $1-x^2-2kt$ of the diffusion equation $u_t - ku_{xx} = 0$. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \le x \le 1, 0 \le t \le T\}$.

2. Page 45 (4) Consider the diffusion equation

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1, 0 < t < \infty, \\ u(0,t) = 0, \quad u(1,t) = 0, & t > 0, \\ u(x,0) = 4x(1-x), & x \in (0,1). \end{cases}$$

   (a) Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.
   (b) Show that $u(x,t) = u(1-x,t)$ for all $t \ge 0$ and $0 \le x \le 1$.
   (c) Use the energy method to show that $\int_0^1 u^2(x,t)dx$ is a strictly decreasing function of t.

3. Page 52 (4) Solve the diffusion equation $u_t = ku_{xx}$ for $x \in (-\infty, \infty)$ and $t > 0$ if $u(x,0) = \phi(x)$ with $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.

4. Page 52 (15) Prove the uniqueness of the diffusion equation with Neumann boundary conditions:

$$\begin{cases} u_t - u_{xx} = f(x,t), & 0 < x < 1, t > 0, \\ u_x(0,t) = g(t), \quad u_x(1,t) = h(t), & t > 0, \\ u(x,0) = \phi(x), & x \in (0,1), \end{cases}$$

   by the energy method.

5. Page 52 (16) Solve the diffusion equation with constant dissipation:

$$\begin{cases} u_t - u_{xx} + bu = 0, & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), & x \in (-\infty, \infty), \end{cases}$$

   where $b > 0$ is a constant. (Hint: make a change of variables $u(x,t) = e^{-bt}v(x,t)$)

6. Page 55 (1) Show that there is no maximum principle for the wave equation. (hint: try to construct a solution in the form $u(x,t) = f(x-ct) + g(x+ct)$ so that the maximum can be achieved not at $t = 0$ but at some $t > 0$)