Math 442 Homework 10: (due April 16, Monday)

1. Page 184 Problem 3: Prove the uniqueness of the Robin problem

$$\begin{cases} \Delta u = f(x), & \text{in } D; \\ \dfrac{\partial u}{\partial n} + a(x)u = h(x), & \text{on } \partial D \end{cases}$$

   by the energy method provided that $a(x) > 0$ on the boundary. That is, after subtracting two solutions $w = u - v$, multiply the Laplace equation for w by w itself and use the divergence theorem.

2. Page 184 Problem 5: Prove the Dirichlet's principle for the Neumann problem. It asserts that among all real-valued functions $w(x)$ on D the quantity

$$E[w] = \frac{1}{2}\int_D |\nabla w(x)|^2 dx - \int_{\partial D} h(x)w(x)dS,$$

   is the smallest for $w = u$, where u is the solution for the Neumann problem

$$\begin{cases} -\Delta u = 0, & \text{in } D, \\ \dfrac{\partial u}{\partial n} = h(x), & \text{on } \partial D. \end{cases}$$

   It is required to assume that the average of the given function $h(x)$ is zero, *i.e.* $\int_D h(x)dx = 0$. (Hint: follow the method in Section 7.1 (Page 183), or my notes)

3. Page 263 Problem 1: Solve the wave equation in the square $S = \{0 < x < \pi, 0 < y < \pi\}$, with homogeneous Neumann conditions on the boundary, and the initial conditions $u(x, y, 0) = 0$, $u_t(x, y, 0) = \sin^2 x$.

4. Page 281 Problem 1: For the Dirichlet problem in a square whose eigenfunctions are given by (Page 279) (3), list the nine smallest distinct eigenvalues. What are their multiplicities?

5. Page 281 Problem 2: Sketch the nodal set of the eigenfunction $v(x, y) = \sin(3x)\sin(y) + \sin(x)\sin(3y)$ in the square $(0, \pi)^2$. (Hint: Use formulas for $\sin(3x)$ and $\sin(3y)$ together with factorization to rewrite it as $v(x, y) = 2\sin x \sin y(3 - 2\sin^2 x - 2\sin^2 y)$.)