Transformations on matrices and tensors

Chi-Kwong Li College of William and Mary

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Connection between matrices and large data set

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• A matrix is an array of numbers

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

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- It is a representation of a data set in a particular way.
- For example, it can be used to record a black and white picture, where the (i, j) entry represents a particular grey scale level.

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$$A = X \begin{bmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{bmatrix} Y^t = s_1 x_1 y_1^t + \dots + s_n x_n y_n^t$$
$$= s_1 x_1 \otimes y_1 + \dots + s_n x_n \otimes y_n.$$

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• If A has rank k, then $s_{k+1} = \cdots = s_n = 0$. We need k + nk + nk = k(2n + 1) data to represent n^2 data.

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- If A has rank k, then $s_{k+1} = \cdots = s_n = 0$. We need k + nk + nk = k(2n + 1) data to represent n^2 data.
- In fact, we can replace those "small" s_j by 0, and then use

$$\tilde{A} = s_1 x_1 y_1^t + \dots + s_r x_r y_r^t,$$

(2r+1)n data, to approximate A.

Determine "simple" transformations on matrices to maintain the quality of the picture with limited storage?

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Theorem [Li and Tsing, 1990]

Let $T: M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be a linear map. The following are equivalent.

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- There is $k \in \{1, ..., n\}$ such that A and T(A) always have the same kth singular value.
- $\bullet\,$ There are two orthogonal matrices P and Q such that T has the form

 $A \mapsto PAQ$ or $A \mapsto PA^tQ$.

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C.K. Li and N.K. Tsing, Linear operators preserving unitarily invariant norms of matrices, Linear and Multilinear Algebra 26 (1990), 119-132.

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Theorem [Frobenius, 1897]

A linear map $T: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies $\det(T(A)) = \det(A) \text{ for all } A \in M_n(\mathbb{C})$ if and only if there are $M, N \in M_n(\mathbb{C})$ with $\det(MN) = 1$ such that (1) T(A) = MAN for all $A \in M_n(\mathbb{C})$, or (2) $T(A) = MA^tN$ for all $A \in M_n(\mathbb{C})$.

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- Basic idea: one can (has to) organize data as a hyper-matrix (order 3 tensors)

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where $u \otimes v \otimes w = [u_i v_j w_k]$ depends on m + n + p data so that data compression can be done by ...

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• One can also consider higher order tensors $A = [a_{i_1 \cdots i_m}]$.

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Connection to Quantum Computing

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- It is of interest to express to express a general state $[v_{i_1i_2i_3}]$ in the bipartite system as a linear combination of tensor states.

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- One can also consider multi-partite states with more systems.

Entanglement and separability

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Problem [Separability/entanglement of quantum states]

Determine whether a density matrix $C \in M_{mn}$ is separable or entangled,

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Problem [Separability/entanglement of quantum states]

Determine whether a density matrix $C \in M_{mn}$ is separable or entangled, i.e., whether there are positive numbers t_1, \ldots, t_k summing up to 1, and density matrices $A_1, \ldots, A_k \in M_m$ and $B_1, \ldots, B_k \in M_n$ such that

$$C = t_1 A_1 \otimes B_1 + \dots + t_k A_k \otimes B_k.$$

Theorem [Friedland, Li, Poon, Sze, 2012]

A linear map $T: M_{mn} \to M_{mn}$ sending the set of entangled states onto itself if and only if there are unitary matrices $U \in M_m$ and $V \in M_n$ such that T has one of the following forms:

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(1)
$$A \otimes B \mapsto \tau_1(A) \otimes \tau_2(B)$$
, or

(2)
$$m = n$$
 and $A \otimes B \mapsto \tau_2(B) \otimes \tau_1(A)$,

where τ_i has the form $X \mapsto U_i^* X U_i$ or $X \mapsto U_i^* X^t U_i$ for some unitary $U_1 \in M_m$ and $U_2 \in M_n$.

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Let m, n, k be positive integers such that $k \leq mn$, and let $T: M_{mn} \to M_{mn}$ be a complex linear map. The following are equivalent.

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- $||A \otimes B||_k = ||T(A \otimes B)||_k$ for all density matrices $A \in M_m$ and $B \in M_n$.
- There are unitary matrices $U, V \in M_{mn}$ such that T has one of the following form:

 $A \otimes B \mapsto U(A \otimes B)V, \quad A \otimes B \mapsto U(A \otimes B^t)V,$

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• Characterize $T: M_{mn} \to M_{mn}$ leaving invariant certain properties or subsets of tensor states.

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