Transformations on matrices and tensors

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- It is a representation of a data set in a particular way.
- For example, it can be used to record a black and white picture, where the (i, j) entry represents a particular grey scale level.

Data storage and data compression

For every matrix $n \times n$ real matrix A, there are $n \times n$ real orthogonal matrices $X = [x_1 \cdots x_n]$, $Y = [y_1 \cdots y_n]$, and nonnegative numbers $s_1 \geq \cdots \geq s_n$,

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- If A has rank k, then $s_{k+1} = \cdots = s_n = 0$. We need $k + nk + nk = k(2n+1)$ data to represent n^2 data.
- In fact, we can replace those "small" s_j by 0, and then use

$$\tilde{A} = s_1 x_1 y_1^t + \cdots + s_r x_r y_r^t,$$

$(2r+1)n$ data, to approximate A.

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C.K. Li and N.K. Tsing, Linear operators preserving unitarily invariant norms of matrices, Linear and Multilinear Algebra 26 (1990), 119-132.

Linear preservers

Theorem [Frobenius,1897]

A linear map $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies

$$\det(T(A)) = \det(A) \text{ for all } A \in M_n(\mathbb{C})$$

if and only if there are $M, N \in M_n(\mathbb{C})$ with $\det(MN) = 1$ such that

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  where $u \otimes v \otimes w = [u_i v_j w_k]$ depends on $m + n + p$ data so that data compression can be done by ...
- One can also consider higher order tensors $A = [a_{i_1 \cdots i_m}]$.

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- One can also consider multi-partite states with more systems.

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Problem [Separability/entanglement of quantum states]

Determine whether a density matrix $C \in M_{mn}$ is separable or entangled, i.e., whether there are positive numbers $t_1, \dots, t_k$ summing up to 1, and density matrices $A_1, \dots, A_k \in M_m$ and $B_1, \dots, B_k \in M_n$ such that

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Preservers of entanglements

Theorem [Friedland, Li, Poon, Sze, 2012]

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(1) $A \otimes B \mapsto \tau_1(A) \otimes \tau_2(B)$, or

(2) $m = n$ and $A \otimes B \mapsto \tau_2(B) \otimes \tau_1(A)$,

where τ_i has the form $X \mapsto U_i^* X U_i$ or $X \mapsto U_i^* X^t U_i$ for some unitary $U_1 \in M_m$ and $U_2 \in M_n$.

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