Decomposition of unitary gates

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Joint work with Diane Pelejo, Rebecca Roberts, Xiaoyan Yin.
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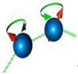
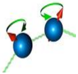
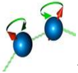
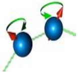
 $\longrightarrow$ Quantum Computing Unit
Optical lattices, NMR, Anyons $\longrightarrow$ 

Quantum bit (Qubit)

Quantum Computing

 $\longrightarrow$ Quantum Computing Unit
Optical lattices, NMR, Anyons $\longrightarrow$ 

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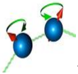
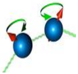
- Store and process information using quantum states (qubits).

Quantum Computing



$\longrightarrow$ **Quantum Computing Unit** — Optical lattices, NMR, Anyons $\longrightarrow$



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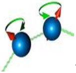

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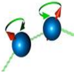

Quantum bit (Qubit)

- Store and process information using quantum states (qubits).
- Apply suitable quantum gates (unitary transformations) to the system
- Apply measurements (unitary transformation) to extract useful information.

Mathematical formulation

Consider a quantum system with two physically measurable states:

$$|\uparrow\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad |\rightarrow\rangle = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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$$|\psi\rangle = a|0\rangle + b|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}, \quad a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1.$$

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- $|0\rangle$ represents a dead cat, $|1\rangle$ represents an alive cat,

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Schrödinger cat interpretation

- $|0\rangle$ represents a dead cat, $|1\rangle$ represents an alive cat,
- $|\psi\rangle = a|0\rangle + b|1\rangle$ represents a cat in the sate of both dead and alive with a probability $|a|^2$ dead and a probability $|b|^2$ alive.

- For two quantum states, $|\psi_1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$, and $|\psi_2\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$, the tensor state of the joint (bipartite) system is represented by

$$|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\psi_2\rangle = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}.$$

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- A general (vector) state is a unit vector in $\mathbb{C}^4$, which is a linear combination of the four measurable states.

A large data set!

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- But, general unitary gates are difficult to generate!
- So, one needs to decompose a general unitary gate to the product of "simple" unitary gates.

A general procedure

- Suppose $U \in M_N$ with $N = 2^n$ is a unitary matrix acting on n-qubit states.

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- Then we have $U = U_1^\dagger \cdots U_k^\dagger$.

A scheme in numerical linear algebra (Givens transform)

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$$U_{41} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \bar{u}_{31}/d_{31} & \bar{u}_{41}/d_{31} \\ & & -u_{41}/d_{31} & u_{31}/d_{31} \end{pmatrix} \text{ so that } U_{41}U = \begin{pmatrix} u_{11} & * & * & * \\ u_{21} & * & * & * \\ d_{31} & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

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$$U_{21} = \begin{pmatrix} \bar{u}_{11} & d_{21} & & \\ -d_{21} & u_{11} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \text{ so that } U_{21}U_{31}U_{41}U = \begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

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Note that the $(1,2), (1,3), (1,4)$ entries will be 0 as well.

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Then consider the second column and construct

$$U_{42} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & * & * \\ & & * & * \end{pmatrix}, \quad U_{32} = \begin{pmatrix} 1 & & & \\ & * & * & \\ & * & * & \\ & & & 1 \end{pmatrix}$$

so that

$$U_{32}U_{42}U_{21}U_{31}U_{41}U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

Let

$$U_{43} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & * & * \\ & & * & * \end{pmatrix}$$

so that

$$U_{43}U_{32}U_{42}U_{21}U_{31}U_{41}U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \det(U) \end{pmatrix} = D.$$

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Thus,

$$U = U_{41}^{\dagger}U_{31}^{\dagger}U_{21}^{\dagger}U_{42}^{\dagger}U_{32}^{\dagger}U_{43}^{\dagger}D.$$

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Number of 2-level matrices used is at most $3 + 2 + 1 = 6$.

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Lemma

Every N-by-N unitary matrix is the product of m 2-level matrices with

$$m \leq (N-1) + \cdots + 1 = \binom{N}{2}.$$

We only need 2-level matrices of the form:

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & v_{11} & v_{12} \\ & & v_{21} & v_{22} \end{pmatrix}, \quad \begin{pmatrix} v_{11} & v_{12} & & \\ v_{21} & v_{22} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & & & \\ & v_{11} & v_{12} & \\ & v_{21} & v_{22} & \\ & & & 1 \end{pmatrix}.$$

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$$\begin{matrix} (00) \\ (01) \\ (10) \\ (11) \end{matrix} \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & v_{11} & v_{12} \\ & & v_{21} & v_{22} \end{pmatrix}, \begin{pmatrix} v_{11} & v_{12} & & \\ v_{21} & v_{22} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & & & \\ & v_{11} & v_{12} & \\ & v_{21} & v_{22} & \\ & & & 1 \end{pmatrix}.$$

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Label the rows and columns of a 4-by-4 unitary matrix by $(00), (01), (10), (11)$.

Let $V = \begin{pmatrix}v_{11} & v_{12} \\ v_{21} & v_{22}\end{pmatrix}$. Then Type 1 and Type 2 matrices correspond to controlled qubit gates changing one qubit, namely,

$$a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$$

to:

$$a_0|00\rangle + a_1|01\rangle + |1\rangle V(a_2|0\rangle + a_3|1\rangle), \qquad (1V) - \text{gate}$$

and

$$|0\rangle V(a_0|0\rangle + a_1|1\rangle) + a_2|10\rangle + a_3|11\rangle, \qquad (0V) - \text{gate.}$$

Other 2-level and controlled single qubit gates

A Type 5 matrix is not so easy to implement because it changes both qubits.

$$\begin{matrix} & (00) & (01) & (10) & (11) & \\ & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & v_{11} & v_{12} & 0 \\ 0 & v_{21} & v_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & & & & \begin{matrix} (00) \\ (01) \\ (10) \\ (11) \end{matrix} \end{matrix}.$$

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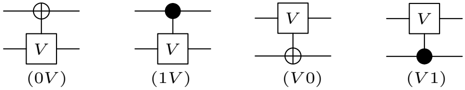
There are two other types of controlled qubit gates on 2 qubits:

$$\begin{matrix} (00) \\ (01) \\ (10) \\ (11) \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & v_{11} & 0 & v_{12} \\ 0 & 0 & 1 & 0 \\ 0 & v_{21} & 0 & v_{22} \end{pmatrix}, \qquad \begin{pmatrix} v_{11} & 0 & v_{12} & 0 \\ 0 & 1 & 0 & 0 \\ v_{21} & 0 & v_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Type 3: (V1)-gate. Type 4: (V0) - gate.

corresponding to $I_2 \otimes |0\rangle\langle 0| + V \otimes |1\rangle\langle 1|$ and $V \otimes |0\rangle\langle 0| + I_2 \otimes |1\rangle\langle 1|$.

The 4 types of controlled qubit gates with the following circuit diagrams:



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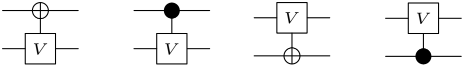

$(0V)$ $(1V)$ $(V0)$ $(V1)$

For $n = 3$, we have fully-controlled qubit gates of the types:

$$(00V), (01V), (10V), (11V), (0V0), (0V1), (1V0), (1V1), (V00), (V01), (V10), (V11).$$

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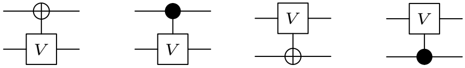

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One easily extends this idea and notation to define fully-controlled gates acting on n-qubits.

$$
\begin{matrix} (00) \\ (01) \\ (10) \\ (11) \end{matrix} \qquad
\overset{\text{Type 1 (1V)}}{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & * & * \\ & & * & * \end{pmatrix}}, \
\overset{\text{Type 2 (0V)}}{\begin{pmatrix} * & * & & \\ * & * & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}, \
\overset{\text{Type 3 (V1)}}{\begin{pmatrix} 1 & & & \\ & * & & * \\ & & 1 & \\ & * & & * \end{pmatrix}}.
$$

Decomposition using only controlled qubit gates

Type 1 (1V) Type 2 (0V) Type 3 (V1)

$$\begin{matrix}(00)\\(01)\\(10)\\(11)\end{matrix} \qquad \begin{pmatrix}1 & & & \\ & 1 & & \\ & & * & * \\ & & * & *\end{pmatrix}, \begin{pmatrix}* & * & & \\ * & * & & \\ & & 1 & \\ & & & 1\end{pmatrix}, \begin{pmatrix}1 & & & \\ & * & & * \\ & & 1 & \\ & * & & *\end{pmatrix}.$$

1. Use Type 1 matrix to make the $(3, 1)$ instead of the $(4, 1)$ entry zero; then use the Type 3 matrix to make the $(4, 1)$ entry zero; then use the Type 2 matrix to make the $(2, 1)$ entry zero.

	Type 1 (1V)	Type 2 (0V)	Type 3 (V1)
(00) (01) (10) (11)	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & * & * \\ & & * & * \end{pmatrix}$,	$\begin{pmatrix} * & * & & \\ * & * & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$,	$\begin{pmatrix} 1 & & & \\ & * & & * \\ & & 1 & \\ & * & & * \end{pmatrix}$.

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2. Use Type 1 matrix to make the $(3, 2)$ instead of the $(4, 2)$ entry zero zero;
   Use Type 3 matrix to make the $(4, 2)$ entry zero.

Decomposition using only controlled qubit gates

	Type 1 (1V)	Type 2 (0V)	Type 3 (V1)

$$\begin{matrix}(00)\\(01)\\(10)\\(11)\end{matrix} \qquad \begin{pmatrix}1 & & & \\ & 1 & & \\ & & * & * \\ & & * & *\end{pmatrix}, \begin{pmatrix}* & * & & \\ * & * & & \\ & & 1 & \\ & & & 1\end{pmatrix}, \begin{pmatrix}1 & & & \\ & * & & * \\ & & 1 & \\ & * & & *\end{pmatrix}.$$

1. Use Type 1 matrix to make the $(3,1)$ instead of the $(4,1)$ entry zero;
   then use the Type 3 matrix to make the $(4,1)$ entry zero;
   then use the Type 2 matrix to make the $(2,1)$ entry zero.

2. Use Type 1 matrix to make the $(3,2)$ instead of the $(4,2)$ entry zero zero;
   Use Type 3 matrix to make the $(4,2)$ entry zero.

3. Use type 1 matrix to make the $(3,4)$ instead of the $(4,3)$ entry zero.

Theorem [Vartiainen et al., 2004]

We can always use single fully controlled single qubit gates to do the decomposition.

A General Result

- In some QC models, one uses the $-3/2, -1/2, 1/2, 3/2$ states of a spin-$1/2$ systems to represent the 2-qubit states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

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- In such a case, it is easier to apply changes between

$$\text{(1)}\quad |00\rangle \text{ and } |01\rangle; \qquad \text{(2)}\quad |01\rangle \text{ and } |10\rangle; \qquad \text{(3)}\quad |10\rangle \text{ and } |11\rangle.$$

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(1) $|00\rangle$ and $|01\rangle$; (2) $|01\rangle$ and $|10\rangle$; (3) $|10\rangle$ and $|11\rangle$.

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In general, let $P = (j_1, j_2, \ldots, j_N)$ be a permutation of $(1, \ldots, N)$.

A P-unitary matrix is a 2-level unitary matrix obtained from I_N by changing its rows and columns indexed by:

$$(j_1, j_2),\ (j_2, j_3),\ \cdots,\ (j_{n-1}, j_n).$$

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Theorem [Li, Roberts, and Yin, 2013]

Let $P = (j_1, j_2, \dots, j_N)$ be a permutation of $(1, 2, \dots, N)$.

Then every N-by-N unitary matrix U can be written as a product of no more than $N(N-1)/2$ P-unitary matrices.

Further simplification

For two qubit system, it is easier to apply the unitary gates of the form:

$$I_2 \otimes V = \begin{pmatrix} V & \\ & V \end{pmatrix} \begin{matrix} (00) \\ (01) \\ (10) \\ (11) \end{matrix} \quad \text{and} \quad V \otimes I_2 = \begin{pmatrix} v_{11} I_2 & v_{12} I_2 \\ v_{21} I_2 & v_{22} I_2 \end{pmatrix}.$$

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They will change the vector states

$$|\psi\rangle = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$$

to:

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$$|\psi\rangle = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$$

to:

$$|0\rangle \otimes V(a_0|0\rangle + a_1|1\rangle) + |1\rangle \otimes (V(a_1|0\rangle + a_2|1\rangle),$$

and

$$V(a_0|0\rangle + a_2|1\rangle) \otimes |0\rangle + (V(a_1|0\rangle + a_3|1\rangle) \otimes |1\rangle.$$

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$$|\psi\rangle = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$$

to:

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and

$$V(a_0|0\rangle + a_2|1\rangle) \otimes |0\rangle + (V(a_1|0\rangle + a_3|1\rangle) \otimes |1\rangle.$$

In many (?) QC models, it is less expensive to implement for some quantum systems.

Further Reduction

Reduction of $U \in M_4$ by 2-C^0V gates and 4-C^1V gates:

$-$			
$1\,(*V)$	$-$		
$3\,(V*)$	$1\,(1V)$	$-$	
$2\,(1V)$	$2\,(V1)$	$1\,(1V)$	$-$

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \to \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \to \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ * & * & * & * \\ 0 & * & * & * \end{pmatrix} \to \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$$

$$\to \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & * & * & * \end{pmatrix} \to \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \to \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

A recursive scheme

Reduction of $U \in M_8$ by 3-C^0V gates, 18-C^1V gates, and 7-C^2V gates:

−							
$1\,(**V)$	−						
$3\,(*V*)$	$1\,(*1V)$	−					
$2\,(*1V)$	$2\,(*V1)$	$1\,(*1V)$	−				
$7\,(V**)$	$3\,(1*V)$	$4\,(1V*)$	$2\,(10V)$	−			
$4\,(1*V)$	$6\,(V*1)$	$3\,(10V)$	$3\,(1V*)$	$1\,(1*V)$	−		
$6\,(1V*)$	$4\,(*1V)$	$5\,(V1*)$	$1\,(1*V)$	$3\,(1V*)$	$1\,(11V)$	−	
$5\,(*1V)$	$5\,(1V*)$	$2\,(1*V)$	$4\,(V11)$	$2\,(11V)$	$2\,(1V1)$	$1\,(11V)$	−

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Annihilate the off-diagonal entries of $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ in columns $1, 2, 3 \ldots$

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- To annihilate the entries in U_{22}, use the same procedures as the previous case with a single control gate in the first qubit (equal to 1).

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Reduction of $U \in M_8$ by 3-C^0V gates, 18-C^1V gates, and 7-C^2V gates:

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- To annihilate the entries in U_{22}, use the same procedures as the previous case with a single control gate in the first qubit (equal to 1).
- A Matlab program was written to do the decomposition.

Counting the control gates

Theorem

Let g_n^k be the number of k-control qubit gates used in our decomposition scheme for an n-qubit unitary gate for $k = 0, 1, \ldots, n-1$.

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4. $g_n^{n-1} = \begin{cases} 1 & \text{if } n = 1 \\ 4 & \text{if } n = 2 \\ 7 + (n-3) & \text{if } n \geq 3 \end{cases}$

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4. $g_n^{n-1} = \begin{cases} 1 & \text{if } n = 1 \\ 4 & \text{if } n = 2 \\ 7 + (n-3) & \text{if } n \geq 3 \end{cases}$
5. $g_n^k = g_{n-1}^k + g_{n-1}^{k-1} + \binom{n-1}{k}$ for all $3 \leq k < n-1$.

A comparison with previous results

A recursion formula was obtained by Vartiainen et al.[1]

$$\mathbf{g}_n^k = \mathbf{g}_{n-1}^k + \mathbf{g}_{n-1}^{k-1} + \max(2^{n-2}, 2^k) + (2^{2n-k-2} - 2^{n-2}) \quad \text{(for } k \geq 1)$$

with the conditions that $\mathbf{g}_m^0 = 2^{m-1}$ for all $m = 1, \ldots, n$.

[1] J. Vartiainen, M. Möttönen, and M. Salomaa, Efficient decomposition of quantum gates, Phys. Rev. Lett. 92 177902 (2004).

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n	g_n^0 / $\mathbf{g}_n^0$	g_n^1 / $\mathbf{g}_n^1$	g_n^2 / $\mathbf{g}_n^2$	g_n^3 / $\mathbf{g}_n^3$	g_n^4 / $\mathbf{g}_n^4$	$T_1(n)$ / $T_2(n)$
1	1 / 1	–	–	–	–	0 / 0
2	2 / 2	4 / 4	–	–	–	4 / 4
3	3 / 4	18 / 14	7 / 10	–	–	32 / 34
4	4 / 8	60 / 50	48 / 40	8 / 22	–	180 / 196
5	5 / 16	180 / 186	242 / 154	60 / 94	9 / 46	880 / 960

[1] J. Vartiainen, M. Möttönen, and M. Salomaa, Efficient decomposition of quantum gates, Phys. Rev. Lett. 92 177902 (2004).

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$$\mathbf{g}_n^k = \mathbf{g}_{n-1}^k + \mathbf{g}_{n-1}^{k-1} + \max(2^{n-2}, 2^k) + (2^{2n-k-2} - 2^{n-2}) \quad (\text{for } k \geq 1)$$

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Starting from $n = 3$, we get a small advantage in our decomposition.

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Starting from $n = 3$, we get a small advantage in our decomposition.
The discrepancy becomes large as n gets larger.
For example, $T_2(10) - T_1(10) = 30,720$.

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In Figure 1, we plot the difference between T_2 and T_1 for n from 1 to 50. We use the log scale in the y-axis.

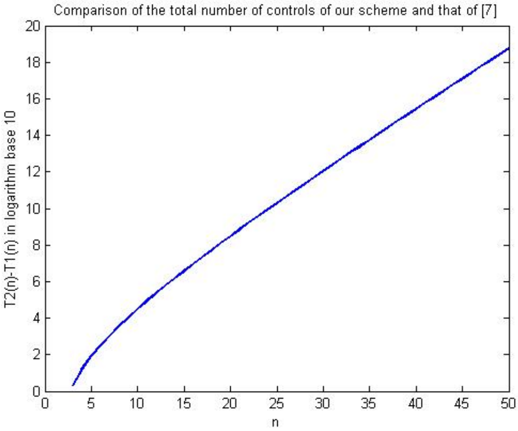

Figure 1

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You are welcomed to talk to me or Diane further if interested!