# Last digit of powers of an integer

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Consider a sequence of integers $2^1$, $2^2$, $2^3$, $2^4$, $\cdots$. Then a repeating pattern for the last digit of $2^n$ occurs: $2^1 : 2$, $2^2 : 4$, $2^3 : 8$, $2^4 : 6$, $2^5 : 2$, $2^6 : 4$, $2^7 : 8$, $\cdots$. In fact, one can see that the pattern repeats after 4 terms: $2, 4, 8, 6, 2, 4, 8, \cdots$.

**Exercise 1**: Try to find the repeating pattern for $3^n$ and $7^n$.

This pattern can be clearly described by using modulo congruence. Recall the definition: $a$ and $b$ are congruent modulo $n$ if $n|(a-b)$, and we write $a \equiv b \pmod{n}$. Then if the last digit of an integer $n$ is $k$, then $k \in \{0,1,2,3,4,5,6,7,8,9\}$, and $n \equiv k \pmod{10}$. Or one can write the decimal expression of an integer $n$ as

$$n = \sum_{i=0}^{m} a_i \cdot 10^i, \quad \text{that is} \quad n = a_m a_{m-1} \cdots a_2 a_1 a_0.$$

Then $a_i \in \{0,1,2,3,4,5,6,7,8,9\}$ ($a_i$ is the $i$-th digit) and $a_m \neq 0$ (highest digit is not zero). Under this notation $n \equiv a_0 \pmod{10}$. The number $a_0$ is called unit digit.

We also recall the basic properties of modulo congruence (Result 4.9-4.11 in textbook): Let $a, b, c, d, k, n \in \mathbb{Z}$ and $n \geq 2$.
(4.9) If $a \equiv b \pmod{n}$, then $ka \equiv kb \pmod{n}$;
(4.10) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$;
(4.11) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.
**Exercise 2**: for $m \in \mathbb{N}$, if $a \equiv b \pmod{n}$, then $a^m \equiv b^m \pmod{n}$. (hint: prove by mathematical induction.)

Now we use modulo congruence to consider the last digit of $1234^{5678}$. This is how it goes:

First $1234 \equiv 4 \pmod{10}$ since $10|(1234-4)$. Next by using Exercise 2, we have $1234^{5678} \equiv 4^{5678} \pmod{10}$. Hence the problem of the last digit of $1234^{5678}$ is reduced to the last digit of $4^{5678}$. We observe the pattern of $4^n$: $4, 16, 64, 256, 1024, \cdots$, so the last digits are $4, 6, 4, 6, 4 \cdots$. Thus the pattern is a simple one: for odd number $n$, the last digit of $4^n$ is 4, and for even number $n$, the last digit of $4^n$ is 6. We prove this fact by using mathematical induction. First we formulate it into a theorem:

**Theorem**. For $n \in \mathbb{N}$, $4^{2n-1} \equiv 4 \pmod{10}$, and $4^{2n} \equiv 6 \pmod{10}$.

*Proof.* When $n = 1$, $4^{2n-1} = 4 \equiv 4 \pmod{10}$, and $4^{2n} = 16 \equiv 6 \pmod{10}$, so the result holds. Suppose the result holds for $n = k$. That is, $4^{2k-1} \equiv 4 \pmod{10}$, and $4^{2k} \equiv 6 \pmod{10}$. Then we prove the result for $n = k+1$. Indeed $4^{2(k+1)-1} = 4^{2k+1} = 4^{2k-1} \cdot 4^2$. From induction assumption: $4^{2k-1} \equiv 4 \pmod{10}$, and obviously $4^2 \equiv 16 \pmod{10}$. Then by using Result 4.11, $4^{2k+1} = 4^{2k-1} \cdot 4^2 \equiv 4 \cdot 16 = 64 \equiv 4 \pmod{10}$. Similarly $4^{2(k+1)} = 4^{2k} = 4^{2k} \cdot 4^2$. From induction assumption: $4^{2k} \equiv 6 \pmod{10}$, and obviously $4^2 \equiv 16 \pmod{10}$. Then by using Result 4.11, $4^{2k+2} = 4^{2k} \cdot 4^2 \equiv 6 \cdot 16 = 96 \equiv 6 \pmod{10}$. Therefore we have proved: $4^{2(k+1)-1} \equiv 4 \pmod{10}$, and $4^{2(k+1)} \equiv 6 \pmod{10}$. From Principle of Mathematical Induction, $4^{2n-1} \equiv 4 \pmod{10}$, and $4^{2n} \equiv 6 \pmod{10}$ for any $n \in \mathbb{N}$. $\square$

Now come back to our problem: since 5678 is even, then the last digit of $4^{5678}$ is 6. Hence the last digit of $1234^{5678}$ is also 6.
**Exercise 3**: Determine the last digit of $8657^{2413}$.