Solve the following problems. Please remember to use complete sentences and good grammar.

1. *(4 Points)* Let $n \in \mathbb{Z}$. Prove that $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$. (Hint: prove $n$ is odd)

2. *(4 Points)* Prove that if $x$ is a real number such that $x^2 + x > 2$, then either $x < -2$ or $x > 1$. (Hint: use axioms and Theorems 1-2 in Notes 1)

3. *(4 Points)* Prove that for every two positive real numbers $a$ and $b$ that

\[(a + b) \cdot \left(\frac{1}{a} + \frac{1}{b}\right) \geq 4.\]

(Hint: use axioms and Theorems 1-2 in Notes 1)

4. *(4 Points)* Let $A, B, C$ be sets. Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.

5. *(4 Points)* Let $A$ and $B$ be sets. Prove that $A = (A - B) \cup (A \cap B)$. (Hint: if $x \in A$, then there are two cases: $x \in B$ or $x \notin B$.)

6. *(4 Points)* Use proof by contradiction in both parts:

   (a) Prove that the product of an irrational number and a nonzero rational number is irrational.

   (b) Prove that there is no smallest positive irrational number.

7. *(4 Points)* Prove by contradiction that if $a$ and $b$ are odd integers, then $4 \nmid (a^2 + b^2)$.

8. *(4 Points)* Consider any three consecutive positive integers. Prove that the cube of the largest cannot be the sum of the cubes of the other two. (Hint: assume that the three consecutive positive integers are $n - 1, n$, and $n + 1$)

9. *(4 Points)* Suppose that $x > 0$ and $x \in \mathbb{R}$. Prove that there exists an irrational number $y$ such that $x < y < 2x$.

   (Hint: Prove in two cases: $x \in \mathbb{Q}$, and $x \notin \mathbb{Q}$, and use the result: (i) the sum of an irrational number and a rational number is irrational; (ii) the product of an irrational number and a nonzero rational number is irrational.)

10. *(extra 2 Points)* Prove that for every three positive real numbers $a$, $b$ and $c$ that

\[(a + b + c) \cdot \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9.\]

11. *(extra 2 Points)* Prove that for every three positive real numbers $a$, $b$ and $c$ that

\[a^2 + b^2 + c^2 \geq ab + bc + ac.\]

Note: For problem 3, 10 and 11, you can only use axioms and Theorems 1-2 in Notes 1, but not other more advanced theorems or known inequalities.