Quiz 2 solution
Math 311

Name: Total Score: 10 pts

1. (3 pts) Define \( a_n = \begin{cases} 
\frac{1}{n+1}, & \text{if } n \text{ is odd;} \\
\frac{n}{n+1}, & \text{if } n \text{ is even.} 
\end{cases} \)

   (a) Find \( \lim \inf a_n \) and \( \lim \sup a_n \); (you do not need to prove)

   (b) Find \( \sup \{ a_n \} \) and \( \inf \{ a_n \} \) (you do not need to prove).

   \( \lim \inf a_n = 0 \) and \( \lim \sup a_n = 1 \); \( \sup \{ a_n \} = 0 \) and \( \inf \{ a_n \} = 1 \).

2. (4 pts) Prove that \( \lim_{n \to \infty} \frac{2n^2 - 1}{n^2 + 1} = 2 \) by using the definition of limit of a sequence.

   For any \( \varepsilon > 0 \), choose \( N = \sqrt{3/\varepsilon} - 1 \). Then when \( n > N \),
   \[ \left| \frac{2n^2 - 1}{n^2 + 1} - 2 \right| = \left| \frac{-3}{n^2 + 1} \right| = \frac{3}{n^2 + 1} < \varepsilon \]
   since \( n > N = \sqrt{3/\varepsilon} - 1 \) is equivalent to \( n^2 + 1 > 3/\varepsilon \). Thus \( \lim_{n \to \infty} \frac{2n^2 - 1}{n^2 + 1} = 2 \).

3. (3 pts) Prove that \( \lim_{n \to \infty} \frac{2n^2 - 1}{n^2 + 1} = 2 \) by using that \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \) and limit theorems.
   Please clearly state which limit theorems you used.

   \[ \frac{2n^2 - 1}{n^2 + 1} = \frac{2 - 1/n^2}{1 + 1/n^2}. \]

   Since \( \lim (1/n^2) = 0 \), then \( \lim (2 - 1/n^2) = 2 \) and \( \lim (1 + 1/n^2) = 1 \) from the limit theorem \( \lim (a_n + b_n) = \lim a_n + \lim b_n \).

   Next from limit theorem \( \lim (a_n/b_n) = \lim a_n / \lim b_n \) if \( \lim b_n \neq 0 \), then \( \lim \frac{2n^2 - 1}{n^2 + 1} = \lim \frac{2 - 1/n^2}{1 + 1/n^2} = \frac{\lim (2 - 1/n^2)}{\lim (1 + 1/n^2)} = 2/1 = 2 \).

4. (extra 1 pt) In homework, we proved that for any two sequences \( (a_n) \) and \( (b_n) \), we must have \( \lim \sup (a_n + b_n) \leq \lim \sup a_n + \lim \sup b_n \). Give an example of \( (a_n) \) and \( (b_n) \) so that \( \lim \sup (a_n + b_n) < \lim \sup a_n + \lim \sup b_n \).

   For example, \( a_n = (-1)^n \) and \( b_n = (-1)^{n+1} \). Then \( \lim \sup a_n = 1 \) and \( \lim \sup b_n = 1 \)
   but \( a_n + b_n = 0 \) for all \( n \), hence \( \lim \sup (a_n + b_n) = 0 \).