(17.8) (a) consider two cases of $f \geq g$ and $f < g$; (b) use $\min(f, g) = (f + g)/2 - |f - g|/2$ and $\max(f, g) = (f + g)/2 + |f - g|/2$; (c) Use 17.3 and 17.4.

(17.9d) Assume that $|x - x_0| \leq 1$, then $|x| \leq |x_0| + 1$, and $|x^2 + x_0x + x_0^2| \leq |x^2| + |x_0x| + |x_0^2| \leq ((|x_0| + 1)^2 + (|x_0| + 1)|x_0| + |x_0|^2 = 3|x_0|^2 + 3|x_0| + 1$. So $|x^3 - x_0^3| = |x - x_0| \cdot |x^2 + x_0x + x_0^2| \leq (3x_0^2 + 3|x_0| + 1)|x - x_0|$. For any $\varepsilon > 0$, select $\delta = \min\{1, \varepsilon/(3|x_0|^2 + 3|x_0| + 1)\}$. Then when $|x - x_0| < \delta$, $|x^3 - x_0^3| = |x - x_0| \cdot |x^2 + x_0x + x_0^2| \leq (3x_0^2 + 3|x_0| + 1)|x - x_0| < \varepsilon$. Hence $g(x) = x^3$ is continuous at $x_0$.

(17.10) (a) Take $x_n = 1/n$, then $\lim x_n = 0$, but $\lim f(x_n) = \lim 1 = 1 \neq 0 = f(0)$, so it is not continuous from Definition 17.1; (b) Take $x_n = (2n\pi + pi/2)^{-1}$, then $\lim x_n = 0$, but $\lim g(x_n) = \lim 1 = 1 \neq 0 = g(0)$, so it is not continuous from Definition 17.1; (c) Take $x_n = 1/n$, then $\lim x_n = 0$, but $\lim \text{sgn}(x_n) = \lim 1 = 1 \neq 0 = \text{sgn}(0)$, so it is not continuous from Definition 17.1; (d) Consider at an integer $p$. Take $x_n = p - 1/n$, then $\lim x_n = p$, but $\lim P(x_n) = \lim 13p + 2 = 13p + 2 \neq 13p + 15 = P(p)$, so it is not continuous at $x = p$ from Definition 17.1.

(17.12a) If $x$ is rational then $f(x) = 0$ from assumption. If $x$ is irrational, then there exists a sequence $(x_n)$ which is rational and $\lim x_n = x$ from 4.7 (the density of rational numbers). Since $f$ is continuous, then $f(x) = \lim f(x_n) = \lim 0 = 0$. So $f(x) = 0$ for all $x \in (a, b)$.

(17.14) For each rational number $x \in \mathbb{R}$, there is a sequence of irrational numbers $(x_n)$ so that $x_n \to x$ as $n \to \infty$, but $f(x_n) = 0$ while $f(x) = 1/q$ for some $q \in \mathbb{Z}$, so $f(x)$ is not continuous at $x$ from Theorem 2.2.

If $x$ is irrational, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n > N$, $1/n < \varepsilon$. There are only finitely many rational numbers $p/q$ in the interval $(x - 1, x + 1)$ with $q \leq N$, and let $\delta$ be the smallest distance from $x$ to any such rational numbers in $(x - 1, x + 1)$. Then for any rational number $y$ in $(x - \delta, x + \delta)$, $f(y) = 1/n < \varepsilon$, and any irrational $y$ in $(x - \delta, x + \delta)$, $f(y) = 0 < \varepsilon$. Then $|f(y) - f(x)| = |f(y)| < \varepsilon$ if $|y - x| < \delta$. This proves the continuity of $f(x)$ at irrational $x$.

(18.4) Consider the function $f(x) = 1/(x - x_0)$. Then it is continuous for $x \neq x_0$. Yet it is unbounded since $\lim x_n = x_0$, and there must be subsequence $x_{n_k}$ which is all larger than or all smaller than $x_0$. In either case, $f(x)$ is unbounded from Theorem 9.10.

(18.5) see back of book

(18.6) Let $f(x) = x - \cos x$. Then $f(0) = -1 < 0$ and $f(\pi/2) = \pi/2 > 0$. $f(x)$ is continuous, so the IVT (18.2) implies that $f(x)$ can take any values between $-1$ and $\pi/2$. In particular, there exists $x_0 \in (0, \pi/2)$ such that $f(x_0) = 0$, then $x_0 = \cos x_0$.

(18.10) Consider $g(x) = f(x + 1) - f(x)$. Then $g(x)$ is continuous. Note that $f(x + 1)$ is continuous since $f(x)$ and $x + 1$ both are. Now $g(0) = f(1) - f(0)$ and $g(1) = f(2) - f(1) = f(0) - f(1) = -[f(1) - f(0)]$. If $f(1) - f(0) = 0$, then $x = 0$ and $y = 1$ are desired solutions. If not, $g(0)$ and $g(1)$ must have opposite signs, then from IVT (18.2), there exists $x_0 \in (0, 1)$ such that $g(x_0) = 0$, then $x = x_0$ and $y = x_0 + 1$ are desired solutions.

(A-17) Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous and it satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that there exists $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x$.

From $f(0) = f(0 + 0) = f(0) + f(0)$, we get $f(0) = 2f(0)$, so $f(0) = 0$. Let $a = f(1)$, then from induction, $f(n) = an$ for all $n \in \mathbb{N}$. From $f(0) = f(n) + f(-n)$, $f(-n) = -an$ for all $n \in \mathbb{N}$. Now for every $p/q \in \mathbb{Q}$, $f(p/q) = pf(1/q)$; and $qf(1/q) = f(1) = a$, so $f(1/q) = 1/q \cdot a$. So we have proved
\[ f(x) = ax \text{ for all } x \in \mathbb{Q}. \] Let \( g(x) = f(x) - ax \). Then \( g(x) = 0 \) for all \( x \in \mathbb{Q} \), and \( g(x) \) is continuous. From Exercise 17.12, \( g(x) = 0 \) for all \( x \in \mathbb{R} \). Thus \( f(x) = ax \) for all \( x \).

(A-18) Prove that there is no continuous function \( f : \mathbb{R} \to \mathbb{R} \) such that, for each \( c \in \mathbb{R} \), the equation \( f(x) = c \) has exactly two solutions.

Suppose, by way of contradiction, that \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function such that, for each \( c \in \mathbb{R} \), the equation \( f(x) = c \) has exactly two solutions. Then there are real numbers \( a < b \) such that \( f(a) = f(b) = 0 \), and \( f(x) \neq 0 \) for all \( x \) such that \( x < a \) or \( a < x < b \) or \( b < x \). Therefore, by the Intermediate Value Theorem, \( f \) cannot take on both positive and negative values on the interval \((-\infty, a)\) (else it would have to take on the value 0 as well), and the same applies to the interval \((a, b)\) and to the interval \((b, \infty)\).

Suppose that \( f \) is positive on the interval \((a, b)\). By Theorem 18.1, \( f \) must be bounded on the closed interval \([a, b]\), and in fact must take on some maximum value, say \( f(c) = M \) for some positive number \( M \), with \( a < c < b \), so that \( f(x) \leq M \) for all \( x \in [a, b] \). Then by the Intermediate Value Theorem, there must be numbers \( x_1 \in (a, c) \) and \( x_2 \in (c, b) \) such that \( f(x_1) = f(x_2) = M/2 \). Moreover, \( f \) does not take on the value \( 2M \) on the interval \([a, b]\) (as \( M \) is an upper bound), so by assumption there must be some \( d \) in \((-\infty, a) \) or \((b, \infty)\) such that \( f(d) = 2M \). Then again by the Intermediate Value Theorem, there must be some \( x_3 \) in \((-\infty, a) \) or \((b, \infty)\) such that \( f(x_3) = M/2 \). (Here \( x_3 \in (d, a) \) if \( d \in (-\infty, a) \), while \( x_3 \in (b, d) \) if \( d \in (b, \infty) \).) In any case, the numbers \( x_1 \), \( x_2 \), and \( x_3 \) must be distinct, because they come from the disjoint intervals \((a, c)\), \((c, b)\), and \((d, a)\) or \((b, d)\), respectively. Since \( f(x_1) = f(x_2) = f(x_3) = M/2 \), this contradicts the assumption on \( f \).