Homework 4 solution
Math 311, Spring 2009

(10.6) (see class discussion)

(10.7) (see back of book)

(10.9) For $s_1 = 1$, $s_{n+1} = \frac{n}{n+1} s_n^2$. Use induction to prove (i) $0 < s_n < 1$; (ii) $s_{n+1} < s_n$. Then \{s_n\} is monotone and bounded, thus $s = \lim s_n$ exists. $s$ satisfies $s = \lim \frac{n}{n+1} s^2 = s^2$, hence $s = 1$ or $s = 0$. But $s_n$ is decreasing and $s_1 = 1$, thus $s \neq 0$. Therefore $s = 0$.

(10.10) For $s_1 = 1$, $s_{n+1} = \frac{1}{3}(s_n + 1)$. Use induction to prove (i) $s_n > 1/2$; (ii) $s_{n+1} < s_n$. Then \{s_n\} is monotone and bounded, thus $s = \lim s_n$ exists. $s$ satisfies $s = \frac{1}{3}/(s + 1)$, so $s = 1/2$.

(10.11) (a) $SL = \{1/n : n \in \mathbb{N}\} \cup \{0\}$. (b) lim sup = 1, lim inf = 0.

(12.1) We prove for lim inf, and the other case is similar. Define $u_N = \inf\{s_n : n \geq N\}$ and $v_N = \inf\{t_n : n \geq N\}$.

Then $(u_N)$ and $(v_N)$ are both nondecreasing. Then from Theorem 10.7, $s = \lim s_n = \lim u_n$ exists and $t = \lim t_n = \lim v_n$ exists. If $s = -\infty$, then $s \leq t$ always holds. So we assume $s > -\infty$. Since $s_n \leq t_n$ for $n \geq N_0$, then $u_n$ is bounded.

Another proof is to use subsequences. From Corollary 11.4, there exists a subsequence $(s_{n_k})$ converging to $\lim s_n = s$, and there exists a subsequence $(t_{n_k})$ converging to $\lim t_n = t$. Since $n_k \to \infty$ when $k \to \infty$, then there exists a $k_0$ such that $n_k > N_0$ when $k > k_0$. Then $s_{n_k} \leq t_{n_k}$ when $k > k_0$. From Exercise 9.9(c), $\lim s_{n_k} \leq \lim t_{n_k}$ and we get $s \leq t$. (this proof seems to be better)

(12.3) $\lim s_n + \lim t_n = 0$, $\lim s_n + \lim t_n = 1$, $\lim s_n + \lim t_n = 2$, $\lim s_n + \lim t_n = 3$, $\lim s_n + \lim t_n = 4$, $\lim s_n + \lim t_n = 0$, $\lim s_n + \lim t_n = 2$.

(12.4) For any $n \geq N$, $s_n \leq s_{n+1} \leq s_{n+2} \leq \cdots$ and $t_n \leq t_{n+1} \leq t_{n+2} \leq \cdots$ for any $n \geq N$. So we can take the sup of the left side to get $\sup\{s_n + t_n : n \geq N\} \leq \sup\{s_n : n \geq N\} + \sup\{t_n : n \geq N\}$. Now define $u_N = \sup\{s_n : n \geq N\}$, $v_N = \sup\{t_n : n \geq N\}$, and $w_N = \sup\{s_n + t_n : n \geq N\}$. Then each of $(u_N)$, $(v_N)$, and $(w_N)$ is monotone (nonincreasing), hence the limit of each exists. Also because $w_n \leq u_n + v_n$, then from Exercise 9.9(c), $\lim w_n \leq \lim(u_n + v_n) = \lim u_n + \lim v_n$, that is $\lim \sup(s_n + t_n) \leq \lim \sup s_n + \lim \sup t_n$.

(12.14) (see class discussion)

(A-10) Prove that if every subsequence of $(s_n)$ has a subsequence which converges to $s$, then the whole sequence $(s_n)$ converges to $s$.

prove by contradiction

(A-12) Let $a > 0$. $x_1 > \sqrt{a}$, and $x_{n+1} = \frac{1}{2} (x_n + \frac{a}{x_n})$, $n = 1, 2, \cdots$. Prove that $(x_n)$ is decreasing, and use Theorem 10.2 to find its limit.

Sketch: First prove that by induction, $1 \leq x_n \leq 2$. Then let $b_n = a_n^2 - 2$, show that $b_{n+1} = \frac{b_n^2}{4a_n^2}$, prove $b_n$ is decreasing, thus $a_n$ is decreasing.

(A-13) Suppose that $s_1 = 0$, $s_{2n} = \frac{s_{2n-1}}{2}$ and $s_{2n+1} = \frac{1}{2} + s_{2n}$. Find $\lim s_n$ and $\lim s_n$.

Notice that $s_{2n} = \frac{1}{4} + \frac{1}{2} s_{2n-2}$, then the subsequence $(s_{2n})$ converges to $1/2$ (monotone and bounded), and $s_{2n+1} = \frac{1}{2} + \frac{1}{2} s_{2n-2}$, then the subsequence $(s_{2n+1})$ converges to $1$ (monotone and bounded). So $SL = \{1/2, 1\}$, and $\lim s_n = 1$ and $\lim s_n = 1/2$. 